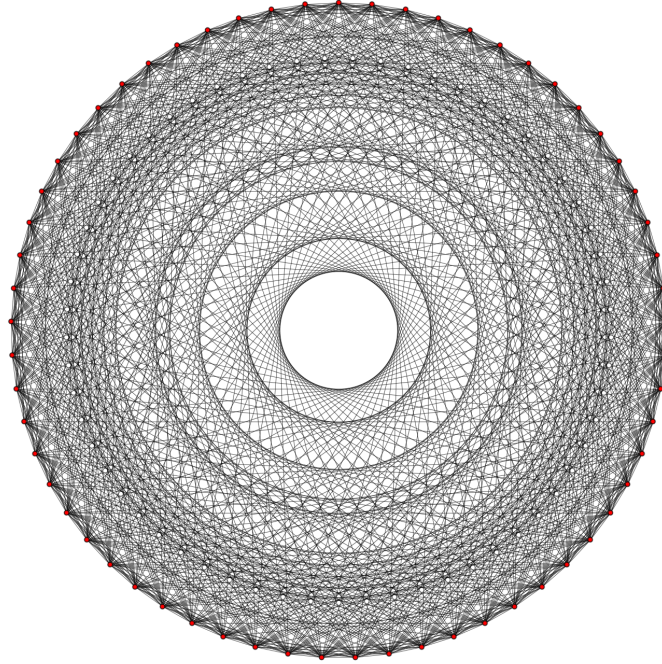


Classification of Parafermionic Invertible Topological Phases of Matter in One-Dimensional Space

by

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ABSTRACT

This thesis develops a classification of one-dimensional invertible topological (IT) and symmetry-protected topological (SPT) phases for lattice systems whose degrees of freedom are parafermions. Parafermions generalize Majorana operators by replacing fermionic $\mathbb{Z}_2$ -grading with a $\mathbb{Z}_N$ structure, leading to exchange relations that are sensitive to ordering and orientation. While parafermions have appeared in various physical contexts, a systematic classification of parafermionic SPT phases has so far been lacking.

We first develop the algebraic foundations of parafermions, including a detailed analysis of the parafermion algebra, its center, and the existence and uniqueness of global parafermion charge operators. A fundamental distinction is established between algebras generated by an even and an odd number of parafermions, which has direct consequences for symmetry realizations and boundary degrees of freedom. Local and global parafermion charge operators are constructed explicitly, and both canonical and generalized notions of charge conjugation are analyzed.

Building on these results, we formulate a framework for parafermionic IT and SPT phases in one dimension. The protecting symmetry group is assumed to contain the global parafermion charge as a normal subgroup, allowing the symmetry to be described as a general group extension. We show that bulk symmetries must act linearly, while boundary zero modes may transform under projective representations, subject to compatibility conditions with the bulk symmetry action. The classification problem is thereby reduced to the classification of projective boundary representations of the protecting symmetry group.

The main result of the thesis is a complete classification of parafermionic SPT phases under the stated assumptions. Even and odd numbers of boundary parafermions lead to qualitatively distinct classification schemes. In the even case, equivalence classes are characterized by ordinary $U(1)$ -valued cohomological data together with an additional index capturing obstructions to linear symmetry implementation. In the odd case, the presence of a nontrivial center requires an extended notion of projective representation involving both $U(1)$ - and $\mathbb{Z}_N$ -valued data. Together, these results generalize the known classification of fermionic SPT phases and provide a unified framework for parafermionic symmetry-protected topology in one dimension.

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⁰The title page depicts the Paley graph [1] of the integer modulo 61 field $\mathbb{F}_{61} \cong \mathbb{Z}/61\mathbb{Z}$. The vertices are distributed radially and denoted by red dots. The edges are represented by black lines. Figure generated using *Wolfram Mathematica*. Each vertex corresponds to a congruence class modulo 61. Let the integers $a \neq b \in \{0, 1, \dots, 60\}$ correspond to two different vertices. There exists an edge between these two vertices if and only if the difference of a and b is a perfect square in $\mathbb{Z}/61\mathbb{Z}$, i.e., there exists a $c \in \{0, 1, \dots, 60\}$ such that $a - b \equiv c^2 \pmod{61}$. As shown in **Proposition 4**, generalized charge operators can realize symmetries if and only if they multiply the parafermion charge by an integer that squares to one modulo the parafermion order. Moreover, such perfect squares can be used to construct isomorphisms between parafermion algebras that respect the locality of the system.

Contents

1	Introduction and overview	9
2	Parafermions and their algebra	19
2.1	Introduction	19
2.2	Definition of the parafermion algebra $\mathcal{P}_N^m$	19
2.3	The centers of parafermionic algebras	22
2.4	Global parafermion charges	26
3	Parafermion algebra with an even number of generators	31
3.1	Introduction	31
3.2	Setup	32
3.3	Local and global parafermion charges	32
3.4	Parafermion charge conjugation	39
3.4.1	Canonical charge conjugation	39
3.4.2	Generalized charge conjugation	41
4	ITPs and SPTs for parafermions	49
4.1	Introduction	49
4.2	One-dimensional chain of parafermions with their symmetries	49
4.3	Hamiltonians, boundary conditions, and boundary algebras	54
4.4	Bulk and boundary representations of the protecting symmetries G^{PF}	55
4.4.1	Linear representations of G^{PF}	58
4.4.2	Projective representations of G^{PF}	58
4.4.3	Compatibility between boundary and bulk representations	61
5	Classification of parafermionic SPTs	63
5.1	Introduction	63
5.2	Setup	63
5.2.1	One-dimensional chain of parafermions	63
5.2.2	Protecting symmetries	64
5.2.3	Hamiltonians and their boundary degrees of freedom	66
5.2.4	Boundary representations when m is even	66
5.2.5	Boundary representations when m is odd	68
5.3	Classification of boundary representations of protecting symmetries	70
5.3.1	Case m -even, i.e., $\mu = 0$	70

5.3.2 Case m -odd, i.e., $\mu = 1$	81
6 Summary of results and outlook	99
A Integer modulo N rings	101
B Group cohomology	107
C Group extensions	111
D Projective representations	119
<i>References</i>	125

Chapter 1

Introduction and overview

The goal of this thesis is to study invertible topological (IT) phases and symmetry-protected topological phases (SPT) for a one-dimensional lattice whose sites host parafermions. The one-dimensional lattice has the topology of a ring if periodic boundary conditions (PBCs) are imposed. It has the topology of an open chain if open boundary conditions (OBCs) are imposed. The adjective topological in this thesis refers to the possibility that there are observable differences when the thermodynamic limit is taken with OBCs instead of PBCs.

An IT phase refers to a family of Hamiltonians constructed out of degrees of freedom¹ assigned to the elements of a set Ω , the base space,² such that

1. any member H of this family
 - (a) is local with respect to some natural distance on base space Ω ,
 - (b) exhibits a nondegenerate gapped ground state after the thermodynamic limit $|\Omega| \uparrow \infty$ has been taken whenever base space Ω has no boundary,³
 - (c) exhibits a gapped ground-state degeneracy after the thermodynamic limit has been taken whenever base space Ω has boundaries,
2. any two members H and H' of this family can be deformed into one another without closing the gap and without leaving the family.

An SPT phase is an IT phase such that all members of the family of Hamiltonians share the same protecting symmetries.

History of IT and SPT phases The first theoretical example of an SPT phase was proposed in 1976 and 1979 [2,3]. The first experimental discovery of an SPT phase took place in 1980 with the observation of the integer quantum Hall effect [4–6]. Both apply to fermionic degrees of freedom. The first example of a bosonic SPT phase took place in 1987 with a remarkable reinterpretation of the antiferromagnetic quantum spin-1 chain [7–11]. It

¹The degrees of freedom can be bosons, fermions, elements of a simple Lie algebra, or objects satisfying some more complicated algebra.

²Base space Ω can be countably finite or it can be a compact subset of some Riemannian manifold.

³We use the notation $|\Omega|$ to denote the cardinality of base space Ω if Ω is countably finite or its volume if it is a compact subset of a Riemannian manifold.

is with the theoretical prediction of the two-dimensional topological insulator in 2004 [12,13] that a systematic classification of IT and SPT phases was undertaken, leading in 2021 to a classification for interacting bosons and interacting fermions obeying on-site symmetries that is believed to be exhaustive [14–16].

Motivating example To make these ideas more concrete and to motivate this thesis, we now review perhaps the simplest IT and SPT phases that can be realized for a one-dimensional lattice. The associated family of Hamiltonians can be traced to one proposed by Kitaev in 2001 that was the first realization of an intrinsic (not protected by symmetry) fermionic IT phase [17]. Consider the family of Hamiltonians⁴

$$H_{r;\mathbf{b}} := \begin{cases} - \sum_{j=1}^{L-\mathbf{b}|r|} i\xi_{j+|r|} \eta_j, & r < 0, \\ + \sum_{j=1}^{L-\mathbf{b}|r|} i\xi_j \eta_j, & r = 0, \\ - \sum_{j=1}^{L-\mathbf{b}|r|} i\xi_j \eta_{j+|r|}, & r > 0. \end{cases} \quad (1.1)$$

Here, each site $j = 1, \dots, L$ hosts the pair ξ_j and η_j of Majorana operators.⁵ All $2L$ Majorana operators obey the Clifford algebra $\mathbb{CL}_{2L}$, namely they all square to the identity, anticommute pairwise, and are Hermitian. The index $\mathbf{b} = 0$ selects PBCs, in which case the site index $j \in \mathbb{Z}_L$. The index $\mathbf{b} = 1$ selects OBCs. The absolute value of the index $r \in \mathbb{Z}$ selects the range of the hopping between two pair of Majorana operators, while its sign selects the direction of the hopping. Each member of the family (1.1) commute with the fermion parity operator

$$U_{\text{fp}} := \prod_{j=1}^L C_{0,j}, \quad C_{0,j} := i\xi_j \eta_j. \quad (1.2)$$

There is no ordering ambiguities between $C_{0,1}, \dots, C_{0,L}$ as they all commute pairwise. The fermion parity U_{fp} squares to the identity and is Hermitian. It generates the fermion-parity symmetry group $\mathbb{Z}_2^{\text{FP}}$. This symmetry group has a very special status. We shall demand that it cannot be broken, neither explicitly nor spontaneously and we shall always work in the 2^{L-1} -dimensional subspace of the 2^L -dimensional Hilbert space on which the eigenvalue of U_{fp} is either $+1$ or -1 . These conditions are examples of fermionic superselection rules.

⁴Kitaev had invented and solved $H_{1;\mathbf{b}}$.

⁵We need two Majorana operators on each site $j = 1, \dots, L$ so that $c_j = (\eta_j + i\xi_j)/2$ and its adjoint $c_j^\dagger = (\eta_j - i\xi_j)/2$ realize a fermionic annihilation and creation operators, respectively. In the fermion representation,

$$H_{0;0} = - \sum_{j=1}^L \left(c_j^\dagger c_j - \frac{1}{2} \right)$$

is the atomic limit of an insulator, while

$$H_{r;\mathbf{b}} = + \sum_{j=1}^{L-\mathbf{b}|r|} \left(c_j^\dagger c_{j+r} + c_j^\dagger c_{j+r}^\dagger + \text{H.c.} \right)$$

are $|r|$ copies of the Kitaev superconducting chain.

The family (1.1) of Hamiltonians is deceptively simple as can be seen by writing [18]

$$H_{r;b} := (-1)^{\delta_{r,0}} \sum_{j=1}^{L-b|r|} C_{r,j}, \quad C_{r,j} := \begin{cases} i\xi_{j+|r|} \eta_j, & r < 0, \\ i\xi_j \eta_j, & r = 0, \\ i\xi_j \eta_{j+|r|}, & r > 0, \end{cases} \quad (1.3)$$

and observing that, for fixed r , any one of $C_{r,1}, \dots, C_{r,L}$ is Hermitian, squares to the identity, and any two of $C_{r,1}, \dots, C_{r,L}$ commute as already stated. On the one hand, $H_{r;b=0}$ is Ising-integrable in that it share the same energy eigenvalues as the classical Ising chain obeying PBCs. As such, it has a nondegenerate gapped ground state when subjected to the superselection rule on the fermion parity. On the other hand, $H_{r;b=1}$ differs from $H_{r;b=0}$ in that it has a $2^{|r|}$ linearly independent degenerate gapped ground states (even before the thermodynamic limit has been taken) as $C_{L-|r|+1}, \dots, C_L$ drop out from $H_{r;b=1}$ relative to $H_{r;b=0}$. This difference would be remarkable and consequential if it were not an artifact of the Ising-integrability of $H_{r;b}$.

One might attribute the $2^{|r|}$ -degeneracy of $H_{r;b=1}$ as an accident stemming from its Ising integrability. To test this hypothesis, we consider first the interpolation [18]

$$H_{0,r;b=1}(\lambda) := (1 - \lambda) H_{r;b=1} + \lambda H_{0;b=1}, \quad 0 \leq \lambda \leq 1. \quad (1.4)$$

Because $H_{r;b=1}$ does not commute with $H_{0;b=1}$, the spectral extensive degeneracies of the classical Ising chain are lifted. Because $H_{0,r;b=1}(\lambda)$ is quadratic, it can be diagonalized by Fourier transformation. One then verifies that the gap closes if and only if $\lambda = 1/2$. When $0 \leq \lambda < 1/2$, the ground-state degeneracy of $H_{0,r;b=1}(\lambda)$ is $2^{|r|}$ -fold. When $1/2 < \lambda \leq 1$, the ground-state of $H_{0,r;b=1}(\lambda)$ is nondegenerate. We conclude that the ground-state degeneracy $2^{|r|}$ of $H_{r;b=1}$ cannot be attributed to its Ising integrability.

To gain more insights on the origin of the ground-state degeneracy $2^{|r|}$ of $H_{r;b=1}$, we observe that this degeneracy arises because the Majorana operators $\eta_1, \dots, \eta_{|r|}$ on the left-hand side of the open chain and the Majorana operators $\xi_{2L-|r|}, \dots, \xi_{2L}$ on the right-hand side of the open chain that are present in $H_{r;b=0}$ do not enter in $H_{r;b=1}$. At this stage, it is essential to recognize that each member of the family (1.1) commutes with the antiunitary operator U_K defined by the conditions

$$U_K i U_K^\dagger = -i, \quad U_K \eta_j U_K^\dagger = +\eta_j, \quad U_K \xi_j U_K^\dagger = -\xi_j, \quad j = 1, \dots, L. \quad (1.5)$$

Hereto, U_K squares to the identity. It generates the symmetry group $\mathbb{Z}_2^K$. Moreover, it commutes with the generators U_{fp} . Hence, each member of the family (1.1) shares the same symmetry group

$$G^{PF} := \mathbb{Z}_2^{FP} \times G, \quad G := \mathbb{Z}_2^K. \quad (1.6)$$

Now, the most general quadratic Hamiltonians for the left and right boundaries are given by

$$H_L^{bd} := \sum_{1 \leq j < l \leq |r|} \lambda_{j,l}^{(L)} i \eta_j \eta_l = -U_K H_L^{bd} U_K^\dagger, \quad \lambda_{j,l}^{(L)} \in \mathbb{R}, \quad (1.7)$$

and

$$H_R^{bd} := \sum_{L-|r| \leq j < l \leq L} \lambda_{j,l}^{(R)} i \xi_j \xi_l = -U_K H_R^{bd} U_K^\dagger, \quad \lambda_{j,l}^{(R)} \in \mathbb{R}, \quad (1.8)$$

respectively. Hence, if we impose that the symmetry G^{PF} holds on the boundary, there is no symmetry-allowed quadratic term that could lift the ground-state degeneracy $2^{|r|}$ of $H_{r;\text{b}=1}$. This argument applies to any Hamiltonian such that the left boundary is spanned by a vector

$$\boldsymbol{\eta}^{\text{T}} = (\eta_1, \dots, \eta_{|r|}) \quad (1.9)$$

of Majorana operators, while the right boundary is spanned by a vector

$$\boldsymbol{\xi}^{\text{T}} = (\xi_{L-|r|}, \dots, \xi_L) \quad (1.10)$$

of Majorana operators. If we declare that the protecting symmetry group (1.6) defines the symmetry class BDI, we conclude that the family of all gapped local Hamiltonians in the symmetry class BDI that are quadratic forms for Majorana operators originating from attaching to each site of a one-dimensional chain a pair of Majorana operators, can be classified by the topological index [19–21]

$$\text{non-interacting-index}(\text{BDI}) = \dots, -1, 0, +1, \dots \cong \mathbb{Z} \quad (1.11)$$

labeling the family (1.1).⁶ The symmetry class BDI host distinct fermionic SPT phases classified by an integer-valued topological index when many-body interactions are excluded.

If we do not impose the symmetry $\mathbb{Z}_2^{\text{K}}$, the quadratic boundary Hamiltonians H_{L}^{bd} and H_{R}^{bd} are allowed by the symmetry $\mathbb{Z}_2^{\text{FP}}$. As quadratic forms for Majorana operators, they can be written as⁷

$$H_{\text{L}}^{\text{bd}} = \boldsymbol{\eta}^{\text{T}} A_{(r)} \boldsymbol{\eta} \quad \text{if } \lambda_{j,l}^{(\text{L})} \in \mathbb{R} = 1, \quad (1.12)$$

and

$$H_{\text{R}}^{\text{bd}} = \boldsymbol{\xi}^{\text{T}} A_{(r)} \boldsymbol{\xi} \quad \text{if } \lambda_{j,l}^{(\text{R})} \in \mathbb{R} = 1, \quad (1.13)$$

respectively. Here, the $|r| \times |r|$ matrix $A_{(r)}$ is antisymmetric with all entries above the diagonal given by the imaginary unit i . When $|r|$ is even, all the eigenvalues of $A_{(r)}$ come in $|r|/2$ pairs ± 1 and the ground-state degeneracy on each boundary is completely lifted. When $|r|$ is odd, $A_{(r)}$ has $(r-1)/2$ pairs of eigenvalues ± 1 and one zero mode, leaving a ground state degeneracy of 2 across the open chain. If we declare that the protecting symmetry group $\mathbb{Z}_2^{\text{FP}}$ defines the symmetry class D, we conclude that the family of all gapped local Hamiltonians in the symmetry class D that are quadratic forms for Majorana operators originating from attaching to each site of a one-dimensional chain a pair of Majorana operators, can be classified by the topological index

$$\text{index}(\text{D}) = 0, 1 \cong \mathbb{Z}_2 \quad (1.14)$$

that labels $H_{r,\text{b}}$ with $r = 0, 1$.

⁶The index is topological because interpolating between two members of the family (1.1) without violating the BDI symmetry and adding interactions is possible if and only if the gap closes.

⁷A term of the form $\boldsymbol{\eta}^{\text{T}} A_{(r)} \boldsymbol{\xi}$ is forbidden by locality, or, more precisely it can only be included with a multiplicative prefactor $\exp(-2vL/\Delta)$ with Δ a characteristic velocity and Δ the spectral gap characterizing the spectrum in the thermodynamic limit.

So far, we have ignored many-body interactions, i.e., monomials of order larger than two in the Majorana operators. For example, terms like

$$H_{\mathbb{L},n}^{\text{bd}} = [\boldsymbol{\eta}^\top A_{(r)} \boldsymbol{\eta}]^{2n}, \quad n = 1, 2, \dots, \quad (1.15)$$

and

$$H_{\mathbb{R},n}^{\text{bd}} = [\boldsymbol{\xi}^\top A_{(r)} \boldsymbol{\xi}]^{2n}, \quad n = 1, 2, \dots, \quad (1.16)$$

respectively, are permissible in the symmetry class BDI. By explicit computation, one verifies that the boundary ground-state degeneracy is completely lifted when $|r|$ is an integer multiple of the number eight. This means that by switching on such boundary interactions, one adiabatically interpolates between $H_{8r,\text{b}}$ and $H_{0,\text{b}}$, i.e., [22–24]

$$\text{non-interacting-index(BDI)} \cong \mathbb{Z} \longrightarrow \text{interacting-index(BDI)} \cong \mathbb{Z}/8\mathbb{Z}. \quad (1.17)$$

Interactions do not change the topological index (1.14) in the symmetry class D as a single Majorana on its boundary cannot interact with itself (it squares to the identity).

By way of explicit constructions of Hamiltonians when space is one-dimensional, we have argued for the existence of two IT phases in the symmetry class D through the representatives $H_{r,\text{b}}$ with $r = 0, 1$, and for the existence of eight SPT phases in the symmetry class BDI through the representatives $H_{r,\text{b}}$ with $r = 0, \dots, 7$. This bottom-up approach has the virtue of providing an intuition, but it lacks generality.

Classification of fermionic IT and SPT phases Is it possible to classify in a systematic way all one-dimensional fermionic ITs and SPT phases by studying one-dimensional chains whose sites each hosts an even number of Majorana operators by simply assuming the existence of boundary zero modes and imposing consistency conditions on the transformations laws they obey under the protecting symmetries, the on-site symmetry group G^{FP} ? This program had been undertaken in 2009 for on-site degrees of freedom that are bosonic in the sense that they commute when they are distinguished by their site index [25,26]. The outcome was a classification of bosonic SPT phases based on the second cohomology group $H^2[G^{\text{BO}}, \text{U}(1)]$ where the protecting symmetries generate the on-site symmetry group G^{BO} . For fermions, the same task is more challenging and was completed 10 years later [27–31].

Goal of thesis The purpose of this thesis is to generalize the boundary-based classification of one-dimensional fermionic IT and SPT phases with a general fermionic on-site symmetry group G^{FP} from Ref. [31] to parafermions (PFs). The point of view taken in Ref. [31] is that, given a fermionic IT phase when base space has boundaries, any obstruction to simultaneously deforming the ground state into a trivial product state while preserving the protecting symmetries, necessarily appears at the boundary and is fully encoded in the symmetry properties of the boundary degrees of freedom.⁸ The central result in Ref. [31] is that any one-dimensional fermionic IT or SPT phase with symmetry G^{FP} is completely characterized by a triplet of indices

$$[(\nu, \rho)], [\mu], \quad (1.18)$$

⁸This is an example of an anomaly inflow [32].

which encode the transformation properties of Majorana boundary degrees of freedom. The $\mathbb{Z}_2$ -valued index $[\mu]$ records the parity of the number of Majorana modes localized on a boundary, distinguishing between intrinsically fermionic phases ($[\mu] = 1$) and phases admitting purely local fermionic Fock spaces on the boundary ($[\mu] = 0$). The remaining data $[(\nu, \rho)]$ classify projective representations of the symmetry group on the boundary and take values in suitable cohomology groups determined by G^{FP} .

Our main result is the construction of a classification in terms of a triplet of indices that also characterizes parafermionic SPT phases with a general parafermionic on-site symmetry group G^{PF} . To the best of our knowledge, this result is new, but why would this result be interesting?

Why parafermions? From the mathematical point of view, the fermion-parity symmetry group $\mathbb{Z}_2^{\text{FP}}$ imposes a $\mathbb{Z}_2$ -graded structure of the Hilbert space that is nothing but the implementation of the Pauli principle through the antisymmetrization of all many-body states in the fermionic many-body Fock space. It is then very natural to generalize this $\mathbb{Z}_2$ -graded structure of the Hilbert space to a $\mathbb{Z}_N$ -graded structure. This is achieved by demanding that each site $j = 1, \dots, L$ hosts the pair ξ_j and η_j of parafermion operators. All $2L$ parafermion operators differ from Majorana operators in that (i) it is their N -th power that is the identity, (ii) their adjoint is their $(N - 1)$ -th power, and (iii) they pick up either one of the two phases

$$\omega_N \equiv \exp(+i2\pi/N) \quad (1.19)$$

or

$$\omega_N^* \equiv \exp(-i2\pi/N) \quad (1.20)$$

when exchanged. When $N = 2$, we recover the Clifford algebra of $2L$ Majorana operators, but when $N > 2$ the mathematics and the physics is enriched. As a set, $\mathbb{Z}_N$ is linearly ordered, but this linear order is not compatible with the Abelian group structure of $\mathbb{Z}_N$. For $\mathbb{Z}_2$, this incompatibility is algebraically harmless because the nontrivial exchange phase is real and orientation-blind,

$$\omega_2 = \omega_2^* = -1, \quad (1.21)$$

and thus unique. For $N > 2$, the cyclic structure

$$1, \quad \omega_N, \quad \dots, \quad \omega_N^{N-1}, \quad (1.22)$$

generates complex, orientation-sensitive phases, making the absence of ordering an intrinsic and unavoidable feature of parafermionic algebras when $N > 2$. Once the protecting on-site symmetry is enlarged from fermion parity $\mathbb{Z}_2^{\text{FP}}$ to a parafermion charge symmetry $\mathbb{Z}_N^{\text{PF}}$ with $N > 2$, the corresponding boundary obstructions can no longer be realized within a Clifford algebra. They necessarily require operators whose exchange relations carry a nontrivial $\mathbb{Z}_N$ phase and whose algebra is sensitive to orientation. Parafermions are precisely the minimal operators that realize such boundary obstructions. From this viewpoint, replacing Majorana by parafermion operators is not optional, but forced upon us by the structure of the symmetry group $\mathbb{Z}_N$. Hereto, how to marry the symmetry group $\mathbb{Z}_N$ with a symmetry group G differs when $N > 2$ compared to $N = 2$. In the case of $N = 2$, conservation of fermion parity

implies the existence of fermionic superselection rules that state that the full symmetry group G^{FP} is a central extension of the fermion-parity subgroup $\mathbb{Z}_2^{\text{FP}}$, i.e.,

$$G = G^{\text{FP}} / \mathbb{Z}_2^{\text{FP}} \quad (1.23)$$

is a group with $\mathbb{Z}_2^{\text{FP}}$ belonging to the center of the full symmetry group G^{FP} . In the case of $N > 2$, conservation of parafermion charge implies the existence of parafermionic superselection rules that state that the full symmetry group G^{PF} is an extension of the parafermion charge subgroup $\mathbb{Z}_N^{\text{PF}}$, i.e.,

$$G = G^{\text{PF}} / \mathbb{Z}_N^{\text{PF}} \quad (1.24)$$

is a group with $\mathbb{Z}_N^{\text{PF}}$ a normal subgroup of the full symmetry group G^{PF} . Had we postulated that the operators ξ_j and η_j generate a parafermion algebra on each site $j = 1, \dots, L$ but commute on different sites, superselection rules would then not be required. If so, the relationship between the bosonic symmetry group $\mathbb{Z}_N^{\text{BO}}$ and the full symmetry group G^{BO} implies that

$$G = G^{\text{BO}} / \mathbb{Z}_N^{\text{BO}} \quad (1.25)$$

is a mere quotient space, i.e., it does not need to be a group. The different ways to embed the bosonic charge $\mathbb{Z}_N$, the fermion parity $\mathbb{Z}_2$, and the parafermion charge $\mathbb{Z}_N$ in the full symmetry group is the reason why the classifications of bosonic, fermionic, and parafermionic IT and SPT phases are different. The bosonic classification is contained in the fermionic one, while the fermionic classification is contained in the parafermionic one.⁹

History of parafermions From a physics point of view, parafermions first appeared in the study of one-dimensional quantum many-body systems as nonlocal but algebraically elementary degrees of freedom that naturally encode order and disorder. Their origin can be traced back to the seminal work of Kadanoff and Fradkin in the early 1980s on the $\mathbb{Z}_N$ generalization of the quantum Ising chain [33].

In that context, parafermions arise through a nonlocal transformation generalizing the Jordan-Wigner transformation [34]. While the Jordan-Wigner transformation maps spin-1/2 operators in a one-dimensional chain to fermionic creation and annihilation operators, the Fradkin-Kadanoff transformation maps $\mathbb{Z}_N$ -clock variables to operators obeying parafermionic exchange relations [33]. Crucially, this construction relies on the linear ordering of sites in one dimension and produces operators whose nontrivial commutation relations are entirely encoded in string operators extending along the chain. In this sense, parafermions are intrinsically tied to one-dimensional geometry and do not admit a purely local realization in higher dimensions.

The physical role of parafermions is to unify order and disorder degrees of freedom within a single algebraic framework. In the $\mathbb{Z}_N$ -clock model, local order parameters and nonlocal disorder operators are related by duality transformations, and parafermions naturally interpolate between them. This mirrors the role played by Majorana fermions in the Ising chain, where unpaired Majorana operators at the boundary signal a topologically nontrivial

⁹Bosons can always be thought of as bound states of fermions. When base space is one-dimensional, fermions can be thought of as bound states of parafermions.

phase. Parafermions generalize this phenomenon to $\mathbb{Z}_N$ -symmetric systems, where boundary zero modes carry fractionalized quantum numbers and obey exchange relations characterized by N -th roots of unity.

From the many-body perspective of the 21st century, one-dimensional parafermionic Hamiltonians have two interpretations [35]. On the one hand, they can be thought of as an emergent description of a one-dimensional interface separating two distinct two-dimensional gapped phases, one of which supports non-Abelian topological order. On the other hand, they can be thought of as stand alone one-dimensional lattice Hamiltonians. In either case, the question arises whether these parafermionic Hamiltonians realize one-dimensional IT or SPT phases [35–46]. In this perspective, parafermions occupy a role in $\mathbb{Z}_N$ -symmetric one-dimensional many-body physics that is directly analogous to the role of Majorana fermions in $\mathbb{Z}_2$ -symmetric systems. They are not introduced by hand, but are forced upon us by the simultaneous requirements of locality, symmetry, and the existence of nontrivial gapped phases with protected boundary structure. This historical and physical perspective motivates their use as natural boundary degrees of freedom in a classification theory of one-dimensional parafermionic IT and SPT phases.

Outline of the thesis This thesis is organized as follows.

Chapter 1 introduces invertible topological (IT) and symmetry-protected topological (SPT) phases in one-dimensional systems, reviews their historical development for bosonic and fermionic degrees of freedom, and motivates the study of parafermions as natural boundary degrees of freedom beyond Majorana fermions. It also summarizes the physical and mathematical goals of the thesis.

Chapter 2 develops the algebraic foundations of parafermions. We define the parafermion algebra, analyze its internal structure, compute its center, and characterize the existence and uniqueness of global parafermion charge operators. A qualitative distinction between even and odd numbers of parafermion generators is established and will play a central role in later chapters.

Chapter 3 specializes the discussion to parafermion algebras with an even number of generators. Local and global parafermion charge operators are constructed explicitly, a canonical $\mathbb{Z}_N$ -graded basis of the Hilbert space is introduced, and both canonical and generalized charge conjugation symmetries are analyzed. The number-theoretic constraints underlying admissible charge conjugations are derived.

Chapter 4 formulates the framework for invertible and symmetry-protected topological phases for one-dimensional parafermion chains. We introduce the class of Hamiltonians under consideration, discuss boundary conditions and boundary zero modes, and motivate the structure of the protecting symmetry group as a group extension of the parafermion charge symmetry. The distinction between linear bulk representations and projective boundary representations of the symmetry group is established.

Chapter 5 contains the main classification results of the thesis. The classification of parafermionic SPT phases is reduced to the classification of projective representations of the protecting symmetry group on the boundary. Even and odd numbers of boundary parafermions are treated separately, leading to distinct cohomological classification schemes and a complete parametrization of equivalence classes of parafermionic SPT phases within

the stated assumptions.

Chapter 6 summarizes the main results and discusses possible extensions and open questions, including generalizations to other symmetry settings and connections to higher-dimensional systems.

The appendices collect background material and technical results used throughout the thesis. **Appendix A** reviews arithmetic over integer modulo rings. **Appendix B** summarizes group cohomology with an emphasis on low-degree cochains and cocycles. **Appendix C** reviews group extensions and their classification. **Appendix D** discusses projective representations and their equivalence.

Chapter 2

Parafermions and their algebra

2.1 Introduction

This chapter develops the algebraic framework for parafermions, which generalize Majorana operators by replacing the usual $\mathbb{Z}_2$ -grading with a $\mathbb{Z}_N$ structure. Building on this idea, we define the parafermion algebra $\mathcal{P}_N^m$ as a noncommutative algebra generated by operators obeying generalized exchange relations, and establish its interpretation as a twisted version of the group algebra of $(\mathbb{Z}/N\mathbb{Z})^m$.

We then analyze the internal structure of this algebra. In particular, we characterize its center by reducing the commutation constraints to a system of linear equations over the integer modulo N ring $\mathbb{Z}/N\mathbb{Z}$, and solve these explicitly, revealing a qualitative distinction between even and odd numbers of generators. This dichotomy plays a central role in the subsequent construction of global parafermion charges.

Finally, we introduce the notion of a global parafermion charge and derive necessary and sufficient conditions for its existence. We show that such charges exist precisely when the number of parafermions is even, and study their uniqueness properties, including how they can be recovered in extended algebras when this condition fails. Together, these results provide a structural understanding of parafermion algebras that will be essential for their application in later chapters.

We define in **Section 2.2** the abstract parafermion algebra $\mathcal{P}_N^m$ and derive some elementary properties. In **Section 2.3** we compute the centers of the parafermion algebra $\mathcal{Z}(\mathcal{P}_N^m)$. We conclude with **Section 2.4**, where we define global parafermion charge operators and characterize their existence.

2.2 Definition of the parafermion algebra $\mathcal{P}_N^m$

Let m and N be a pair of non-vanishing positive integers. Let

$$\omega_N := e^{\frac{2\pi i}{N}} \tag{2.1a}$$

denote the complex-valued N -th root of unity. Let

$$\text{sgn}(x) := \begin{cases} +1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases} \quad (2.1b)$$

denote the sign function.

Definition 1 (Parafermions and their algebra). Given are m linear operators $(\gamma_1, \dots, \gamma_m)$ and their adjoints $(\gamma_1^\dagger, \dots, \gamma_m^\dagger)$ acting on the Hilbert space $\mathfrak{H}$ with the properties

$$\gamma_i^N = \mathbb{1}, \quad (2.2a)$$

$$\gamma_i \gamma_{i'} = \omega_N^{\text{sgn}(i' - i)} \gamma_{i'} \gamma_i, \quad (2.2b)$$

$$\gamma_i^\dagger = \gamma_i^{N-1}, \quad (2.2c)$$

for $i, i' = 1, \dots, m$. The algebra $\mathcal{P}_N^m$ is the N^m -dimensional vector space over the field of complex numbers

$$\mathcal{P}_N^m := \text{span}_{\mathbb{C}} \left\{ \gamma_1^{k_1} \cdots \gamma_m^{k_m} \mid k_1, \dots, k_m = 0, 1, \dots, N-1 \right\} \quad (2.2d)$$

equipped with the bilinear product defined through Eq. (2.2b). Each one of the m operators $(\gamma_1, \dots, \gamma_m)$ is called a parafermion and the algebra $\mathcal{P}_N^m$ denotes the parafermion algebra generated by m parafermions of order N .

Remark 1 (Parafermions via integer modulo N rings). Equation (2.2a) implies that each parafermion operator is of order N . Correspondingly, we can interpret the exponent k_i of the operators γ_i entering the basis element $\gamma_1^{k_1} \cdots \gamma_m^{k_m}$ as an element of the ring of integers modulo N , i.e.,

$$\mathbb{Z}/N\mathbb{Z} = \{0, 1, \dots, N-1\}, \quad (2.3)$$

and we can identify the basis element $\gamma_1^{k_1} \cdots \gamma_m^{k_m}$ with the element

$$\mathbf{k}^\top = (k_1, \dots, k_m) \in (\mathbb{Z}/N\mathbb{Z})^m. \quad (2.4)$$

The dimensionality N^m of $\mathcal{P}_N^m$, when interpreted as a vector space over the field $\mathbb{C}$, is then that of $(\mathbb{Z}/N\mathbb{Z})^m$. A review of the arithmetics of integers modulo N and the ring $\mathbb{Z}/N\mathbb{Z}$ is presented in **Appendix A**.

Remark 2 (Formal constructions of parafermionic algebras). Equation (2.2b) implies that changing the order of two consecutive parafermion operators γ_i and $\gamma_{i'}$ in a product introduces a $U(1)$ -valued multiplicative phase factor that depends on their relative position $i' - i$, instead of the multiplicative phase factor ± 1 produced by ordinary commutation or anticommutation. As $\mathcal{P}_N^m$ is not just a vector space, but is also equipped with the bilinear product corresponding to operator multiplication, the product of two arbitrary elements corresponding to the m -tuples $\mathbf{k}, \mathbf{l} \in (\mathbb{Z}/N\mathbb{Z})^m$ obey

$$\gamma^{\mathbf{k}} \cdot \gamma^{\mathbf{l}} \propto \gamma^{\mathbf{k}+\mathbf{l}}, \quad (2.5a)$$

where we have introduced the short-hand notation

$$\gamma^{\mathbf{k}} \equiv \gamma_1^{k_1} \dots \gamma_m^{k_m}, \quad \gamma^{\mathbf{l}} \equiv \gamma_1^{l_1} \dots \gamma_m^{l_m}. \quad (2.5b)$$

Such a product thus realizes the standard group addition multiplication in $(\mathbb{Z}/N\mathbb{Z})^m$, provided we ignore the $U(1)$ -valued proportionality factor occurring when reordering the parafermions to bring the left-hand side to the right-hand side of Eq. (2.5a). Thus, $\mathcal{P}_N^m$ is a so-called twisted group algebra of $(\mathbb{Z}/N\mathbb{Z})^m$ over the field $\mathbb{C}$ [47]. It will turn out to be useful to view the elements of the parafermion algebra as polynomials over the m variables $\{\gamma_i\}_{i=1}^m$, whence we write

$$\mathcal{P}_N^m \equiv \mathbb{C}[\gamma_1, \gamma_2, \dots, \gamma_m]. \quad (2.6)$$

Formally, the right-hand-side of Eq. (2.6) may be constructed as a multivariate skew polynomial ring [48] quotiented by the ideal generated by the terms $\{\gamma_i^N - 1\}_{i=1}^m$ due to the parafermions being of order N .

Remark 3. Equation (2.2c) implies that each parafermion is a unitary operator acting on the Hilbert space $\mathfrak{H}$.

Example 1 (Majorana algebra, $N = 2$, $m \geq 1$). Consider the parafermion algebra $\mathcal{P}_2^{m \geq 1}$. Since $\omega_2 = -1$, it follows that the parafermionic operators anticommute pairwise. As they are unitary operators of order 2, they also are Hermitian operators. Parafermion operators of order two are called Majorana operators and the algebra $\mathcal{P}_2^{m \geq 1}$ they generate coincides with the Clifford algebra $\mathbb{C}\mathbb{L}_{m \geq 1}$. Whenever, m is even, the Majorana operators can be arranged into $m/2$ ordered pairs. Each ordered pair of Majorana operators defines a pair of creation and annihilation fermion operators. These $m/2$ pairs of creation and annihilation operators generate a $2^{m/2}$ -dimensional fermionic Fock space [31].

Example 2 (Clock algebra, $N \geq 2$, $m = 2$). Consider the parafermion algebra $\mathcal{P}_{N \geq 2}^2$. This algebra can be explicitly represented by the pair of matrices

$$\gamma_1 = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega_N & 0 & \dots & 0 \\ 0 & 0 & \omega_N^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \omega_N^{N-1} \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}. \quad (2.7)$$

The matrix γ_1 is diagonal with N distinct eigenvalues. The matrix γ_2 is off-diagonal in such a way that it maps the n -th eigenstate of γ_1 to the $(n+1)$ -th eigenstate of γ_1 in a cyclic fashion. Hence, the parafermionic algebra $\mathcal{P}_N^2$ realizes the algebra of a pair of order- N clock operators, see Ref. [39] for a study of symmetry protected topological phases in bosonic clock models.

Remark 4. Parafermions can be thought of as generalizations of either fermions from the point of view of **Example 1** or clock models from the point of view of **Example 2**.

Lemma 1 (Exchange of two basis elements from $\mathcal{P}_N^m$). Given is the parafermionic algebra $\mathcal{P}_N^m$ with $N \geq 2$ and $m \geq 1$. For any two basis elements

$$\gamma^{\mathbf{k}} \equiv \gamma_1^{k_1} \cdots \gamma_m^{k_m}, \quad \mathbf{k}^\top = (k_1, \dots, k_m) \in (\mathbb{Z}/N\mathbb{Z})^m, \quad (2.8a)$$

$$\gamma^{\mathbf{l}} \equiv \gamma_1^{l_1} \cdots \gamma_m^{l_m}, \quad \mathbf{l}^\top = (l_1, \dots, l_m) \in (\mathbb{Z}/N\mathbb{Z})^m, \quad (2.8b)$$

of $\mathcal{P}_N^m$, we have the exchange relation

$$\gamma^{\mathbf{k}} \gamma^{\mathbf{l}} = \omega_N^{\Phi(\mathbf{k}, \mathbf{l})} \gamma^{\mathbf{l}} \gamma^{\mathbf{k}}, \quad \Phi(\mathbf{k}, \mathbf{l}) := \sum_{i,j=1}^m \text{sgn}(j-i) k_i l_j. \quad (2.8c)$$

Proof. Repeated application of Eq. (2.2b). $\square$

Example 3 (Exchange with individual parafermions). **Lemma 1** applied to the case

$$l_1 = \cdots = l_{j-1} = 0, \quad l_j = 1, \quad l_{j+1} = \cdots = l_m = 0, \quad (2.9)$$

simplifies to

$$\gamma^{\mathbf{k}} \gamma_j = \omega_N^{\phi_j(\mathbf{k})} \gamma_j \gamma^{\mathbf{k}}, \quad \phi_j(\mathbf{k}) := \sum_{i=1}^m \text{sgn}(j-i) k_i. \quad (2.10)$$

2.3 The centers of parafermionic algebras

Definition 2 (Center $\mathcal{Z}(\mathcal{P}_N^m)$ of $\mathcal{P}_N^m$). The center $\mathcal{Z}(\mathcal{P}_N^m)$ of the parafermionic algebra $\mathcal{P}_N^m$ is the subalgebra consisting of all elements of $\mathcal{P}_N^m$ that commute with every element of $\mathcal{P}_N^m$.

The center $\mathcal{Z}(\mathcal{P}_N^m)$ is a nonempty set as the identity

$$\mathbb{1} \equiv \gamma^0, \quad 0 = (0, \dots, 0) \in (\mathbb{Z}/N\mathbb{Z})^m, \quad (2.11)$$

commutes with all elements of $\mathcal{P}_N^m$. For any $\mathbf{k} \in (\mathbb{Z}/N\mathbb{Z})^m \setminus \{0\}$, the basis element $\gamma^{\mathbf{k}}$ of $\mathcal{P}_N^m$ when interpreted as a vector space belongs to the center $\mathcal{Z}(\mathcal{P}_N^m)$ if and only if $\gamma^{\mathbf{k}}$ commutes with all parafermions γ_j with $j = 1, \dots, m$. By **Example 3**, this means that k must fulfill the set of equations

$$\phi_j(\mathbf{k}) = \sum_{i=1}^m \text{sgn}(j-i) k_i \stackrel{!}{=} 0 \in \mathbb{Z}/N\mathbb{Z}, \quad \forall j = 1, \dots, m. \quad (2.12)$$

This is a system of m linear equations in m variables over the ring $\mathbb{Z}/N\mathbb{Z}$. It is equivalent to the m congruences

$$\phi_j(\mathbf{k}) = \sum_{i=1}^m \text{sgn}(j-i) k_i \stackrel{!}{\equiv} 0 \pmod{N}, \quad \forall j = 1, \dots, m. \quad (2.13)$$

We have thus proved the following lemma.

Lemma 2 (Condition for $\gamma^{\mathbf{k}} \in \mathcal{Z}(\mathcal{P}_N^m)$). Given are the parafermionic algebra $\mathcal{P}_N^m$ with $N \geq 2$ and $m \geq 1$ and $\mathbf{k} \in (\mathbb{Z}/N\mathbb{Z})^m$. The basis element

$$\gamma^{\mathbf{k}} \equiv (\gamma_1^{k_1} \dots \gamma_m^{k_m}) \quad (2.14a)$$

belongs to the center $\mathcal{Z}(\mathcal{P}_N^m)$ of the parafermion algebra $\mathcal{P}_N^m$ if and only if

$$A_{(m)} \mathbf{k} = \mathbf{0}, \quad \mathbf{0} = (0, \dots, 0)^\top, \quad (2.14b)$$

where $A_{(m)}$ when $m > 1$ is the $\mathbb{Z}/N\mathbb{Z}$ -valued matrix

$$A_{(m)} := \begin{pmatrix} 0 & -1 & -1 & -1 & \dots & -1 \\ 1 & 0 & -1 & -1 & \dots & -1 \\ 1 & 1 & 0 & -1 & \dots & -1 \\ 1 & 1 & 1 & 0 & \dots & -1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 0 \end{pmatrix} \in \text{Mat}_{m \times m}(\mathbb{Z}/N\mathbb{Z}) \quad (2.14c)$$

and $A_{(m)}$ when $m = 1$ is the matrix

$$A_{(1)} = 0 \in \mathbb{Z}/N\mathbb{Z}. \quad (2.14d)$$

Remark 5 (Tournament matrix). As a warmup, let us interpret the matrix $A_{(m)}$ defined in Eq. (2.14c) as having entries over the standard (algebraic) field $\mathbb{R}$. The matrix $A_{(m)}$ is then an example of a tournament matrix. As such it obeys

$$\det(A_{(m)}) = 0 \iff m \text{ is odd}, \quad (2.15a)$$

and

$$\dim(\ker(A_{(m)})) = \begin{cases} 0, & \text{if } m \text{ is even,} \\ 1, & \text{if } m \text{ is odd,} \end{cases} \quad (2.15b)$$

see [49].

While it is tempting to solve Eq. (2.14b) by linear algebraic means (as is done in *Remark 5*), special care must be taken due to the fact that the variables and coefficients take values in the integer modulo N ring $\mathbb{Z}/N\mathbb{Z}$. As a ring, $\mathbb{Z}/N\mathbb{Z}$ has less structure than a field (not to be confused with a scalar or vector field, cf. **Appendix A** for definitions). In fact, $\mathbb{Z}/N\mathbb{Z}$ is a field if and only if N is a prime number. For example, for $N = 8$, one has $2 \cdot 4 = 8 \equiv 0 \pmod{4}$, i.e., there are two non-zero elements in $\mathbb{Z}/4\mathbb{Z}$ whose product is 0. These are called zero-divisors and do not exist in fields. Moreover, recall that a ring is a set with an Abelian group structure (e.g., addition) combined with a monoidal structure (e.g., multiplication), such that both operations are compatible (e.g., distributive). Monoidal means that we have an associative product with an identity element (the multiplicative unit 1), but potentially lacking inverse elements. In simpler terms, “division” may be ill-defined, which in turn means that techniques from linear algebra such as Gaussian elimination may fail.

For this reason, we opt to compute the center by using elementary means that avoid division in $\mathbb{Z}/N\mathbb{Z}$. As is the case for the simpler case of *Remark 5*, the center will only depend on the parity of the number of parafermion operators m and, in particular, is independent of the parafermion order N .

Proposition 1 (Construction of $\mathcal{Z}(\mathcal{P}_N^m) \subset \mathcal{P}_N^m$). Given is the parafermion algebra $\mathcal{P}_N^m$ with $m \geq 1$ and $N \geq 2$. If $m > 1$ is even, then the center is one-dimensional,

$$\mathcal{Z}(\mathcal{P}_N^m) = \text{span}_{\mathbb{C}} \{ \mathbb{1} \}. \quad (2.16a)$$

If $m > 1$ is odd, then the center is N -dimensional,

$$\begin{aligned} \mathcal{Z}(\mathcal{P}_N^m) &= \text{span}_{\mathbb{C}} \left\{ \prod_{i=1}^m \gamma_i^{k_i} \mid \mathbf{k} = (s, -s, \dots, s, -s, s)^{\top}, \quad s \in \mathbb{Z}/N\mathbb{Z} \right\} \\ &\equiv \text{span}_{\mathbb{C}} \{ \mathbb{1}, Y, Y^2, \dots, Y^{N-1} \}, \end{aligned} \quad (2.16b)$$

where

$$Y := \lambda_N \left(\gamma_1 \gamma_2^{\dagger} \right) \cdots \left(\gamma_{m-2} \gamma_{m-1}^{\dagger} \right) \equiv \lambda_N \left(\prod_{j=1}^{\frac{m-1}{2}} \gamma_{2j-1} \gamma_{2j}^{\dagger} \right) \gamma_m \quad (2.16c)$$

is the order- N generator of the center. Here, the multiplicative normalization factor $\lambda_N \in \mathbb{C}$ is chosen such that

$$Y^N \equiv YY^{\dagger} = \mathbb{1}. \quad (2.16d)$$

If $m = 1$, then

$$\mathcal{Z}(\mathcal{P}_N^1) = \mathcal{P}_N^1 = \text{span}_{\mathbb{C}} \{ \mathbb{1}, \gamma \}. \quad (2.16e)$$

Proof. We prove the statement by induction over $m \geq 1$. Note that we must consider the even and odd cases separately. By **Lemma 2**, it suffices to compute the kernels of $A_{(m)}$. The elements of the kernel can then be identified with the parafermionic operators spanning the center of the algebra. We will do so by Gaussian elimination.

Base cases $m = 1$ and $m = 2$ For $m = 1$, we have

$$A_{(1)} = 0, \quad (2.17)$$

and one sees directly that

$$\ker(A_{(1)}) = \{ s \mid s \in \mathbb{Z}/N\mathbb{Z} \}. \quad (2.18)$$

For $m = 2$ one has,

$$A_{(2)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (2.19)$$

and one may directly compute

$$\ker(A_{(2)}) = \{ (0, 0)^{\top} \}. \quad (2.20)$$

Induction over m Suppose the statement holds for $m \geq 3$ (regardless whether even or odd). We compute the kernel of the matrix

$$A_{(m+1)} \in \text{Mat}_{(m+1) \times (m+1)}(\mathbb{Z}/N\mathbb{Z}). \quad (2.21)$$

To perform Gaussian elimination, we may write the matrix equation

$$A_{(m+1)} \mathbf{k} = \mathbf{0}, \quad \mathbf{0}^{\top} \in (\mathbb{Z}/N\mathbb{Z})^{m+1} \quad (2.22)$$

in its augmented form

$$\left[\begin{array}{cccccc|c} 0 & -1 & -1 & -1 & \cdots & -1 & 0 \\ 1 & 0 & -1 & -1 & \cdots & -1 & 0 \\ 1 & 1 & 0 & -1 & \cdots & -1 & 0 \\ 1 & 1 & 1 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 \end{array} \right]. \quad (2.23a)$$

Recall that we may now interchange two rows, multiply a row by a non-zero scalar, or add a scalar multiple of one row to another. Note that we may not necessarily be able to divide a row by a scalar, as the multiplicative inverse need not exist in $\mathbb{Z}/N\mathbb{Z}$.

We begin by multiplying the second row by -1 (i.e., the additive inverse of the multiplicative unit 1), before interchanging it with the first row. This yields

$$\left[\begin{array}{cccccc|c} -1 & 0 & 1 & 1 & \cdots & 1 & 0 \\ 0 & -1 & -1 & -1 & \cdots & -1 & 0 \\ 1 & 1 & 0 & -1 & \cdots & -1 & 0 \\ 1 & 1 & 1 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 \end{array} \right]. \quad (2.23b)$$

Adding the second row to the first then gives

$$\left[\begin{array}{cccccc|c} -1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & -1 & -1 & \cdots & -1 & 0 \\ 1 & 1 & 0 & -1 & \cdots & -1 & 0 \\ 1 & 1 & 1 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 0 & 0 \end{array} \right]. \quad (2.23c)$$

We may now iteratively add the first row to all the rows below the second one. This leads to

$$\left[\begin{array}{cccccc|c} -1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & -1 & -1 & \cdots & -1 & 0 \\ 0 & 0 & 0 & -1 & \cdots & -1 & 0 \\ 0 & 0 & 1 & 0 & \cdots & -1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & 1 & \cdots & 0 & 0 \end{array} \right]. \quad (2.23d)$$

Observe that the lower-right $m - 1$ by $m - 1$ square matrix is nothing but $A_{(m-1)}$. The de-augmented matrix may now be written as

$$\left(\begin{array}{cc|cccc} -1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & -1 & -1 & \cdots & -1 \\ \hline 0 & 0 & & & & \\ 0 & 0 & & & & \\ \vdots & \vdots & & & & \\ 0 & 0 & & & & \end{array} \right). \quad (2.23e)$$

$A_{(m-1)}$

This is to say that we are in a position to use our induction hypothesis.

If $m + 1$ is even, then so is $m - 1$. Thus, an element $\mathbf{k}_{\text{even}}$ of the kernel is always of the form

$$\mathbf{k}_{\text{even}} = (x, y, 0, \dots, 0)^\top. \quad (2.24)$$

We deduce from the first row of the augmented matrix (2.23d) that $x = -y$ must hold. Moreover, the second row implies that $y = 0$. We conclude that

$$\mathbf{k}_{\text{even}} = (0, 0, 0, \dots, 0)^\top. \quad (2.25)$$

If $m + 1$ is odd, then so is $m - 1$. Thus, an element $\mathbf{k}_{\text{odd}}$ of the kernel is always of the form

$$\mathbf{k}_{\text{odd}} = (x, y, s, -s, s, \dots, s, -s, s)^\top, \quad s \in \mathbb{Z}/N\mathbb{Z}. \quad (2.26)$$

We again deduce from the first row of the augmented matrix (2.23d) that $x = -y$ must hold. However, the second row now implies that $y = -s$ (due to $m - 1$ being odd, all pairs of s and $-s$ cancel each other out, leaving only $-s$ at the end). We conclude that

$$\mathbf{k}_{\text{odd}} = (s, -s, s, \dots, s, -s, s)^\top, \quad \forall s \in \mathbb{Z}/N\mathbb{Z}. \quad (2.27)$$

□

2.4 Global parafermion charges

Definition 3 (Global parafermion charge). Given is the parafermion algebra $\mathcal{P}_N^m$ with the integers $m \geq 1$ and $N \geq 2$. A unitary parafermion operator $U_{\text{pf}} \in \mathcal{P}_N^m$ that satisfies

$$U_{\text{pf}} \gamma_j U_{\text{pf}}^\dagger \stackrel{!}{=} \omega_N \gamma_j, \quad j = 1, \dots, m, \quad (2.28)$$

is called a global parafermionic charge.

Lemma 3 (Condition for the existence of global parafermion charge). Given is the parafermion algebra $\mathcal{P}_N^m$ with the integers $m \geq 1$ and $N \geq 2$. A unitary parafermionic operator of the form

$$\gamma^{\mathbf{k}} \equiv \gamma_1^{k_1} \dots \gamma_m^{k_m}, \quad \mathbf{k}^\top \in (\mathbb{Z}/N\mathbb{Z})^m, \quad (2.29a)$$

is a global parafermionic charge if and only if

$$A_{(m)} \mathbf{k} = \mathbf{1}, \quad (2.29b)$$

where the matrix $A_{(m)}$ is defined in **Lemma 2** and

$$\mathbf{1} = (1, \dots, 1)^\top. \quad (2.29c)$$

Proof. The proof follows from an application of **Lemma 1**. □

Proposition 2 (Existence of global parafermion charge). Given is the parafermion algebra $\mathcal{P}_N^m$ with the integers $m \geq 1$ and $N \geq 2$.

- If m is an even integer, then

$$U_{\text{pf}} = (\gamma_1 \gamma_2^\dagger) \cdots (\gamma_{m-1} \gamma_m^\dagger) \equiv \prod_{j=1}^{m/2} (\gamma_{2j-1} \gamma_{2j}^\dagger) \quad (2.30)$$

is a global parafermion charge.

- If m is an odd integer, then a global parafermion charge does not exist in $\mathcal{P}_N^m$.

Proof. We prove the statement by induction over $m \geq 1$. We must consider the even and odd cases separately. By **Lemma 1**, we must solve the inhomogeneous matrix equation

$$A_{(m)} \mathbf{k} = \mathbf{1}, \quad \mathbf{1}^\top \in (\mathbb{Z}/N\mathbb{Z})^m. \quad (2.31)$$

[Recall that when constructing the center of the parafermion algebra, we were solving for the kernel of $A_{(m)}$, which is the homogeneous version of Eq. (2.31).] This set of linear equations is solved by Gaussian elimination.

Base cases $m = 1$ and $m = 2$ For $m = 1$, we have

$$A_{(1)} = 0. \quad (2.32a)$$

There is no $\mathbb{Z}_N$ -valued solution to

$$A_{(1)} k = 1. \quad (2.32b)$$

For $m = 2$, one has

$$A_{(2)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (2.33a)$$

One verifies that the unique solution to Eq. (2.31) is

$$\mathbf{k} = (1, -1)^\top. \quad (2.33b)$$

Induction over m Suppose that the statement holds for $m \geq 3$ (regardless of whether m is even or odd). We solve

$$A_{(m+1)} \mathbf{k} = \mathbf{1}, \quad \mathbf{1}^\top \in (\mathbb{Z}/N\mathbb{Z})^{m+1}. \quad (2.34)$$

To perform Gaussian elimination, we write this matrix equation in its augmented form

$$\left[\begin{array}{cccccc|c} 0 & -1 & -1 & -1 & \cdots & -1 & 1 \\ 1 & 0 & -1 & -1 & \cdots & -1 & 1 \\ 1 & 1 & 0 & -1 & \cdots & -1 & 1 \\ 1 & 1 & 1 & 0 & \cdots & -1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 0 & 1 \end{array} \right]. \quad (2.35a)$$

We begin by multiplying the second row by -1 before interchanging it with the first row. This yields

$$\left[\begin{array}{cccccc|c} -1 & 0 & 1 & 1 & \cdots & 1 & -1 \\ 0 & -1 & -1 & -1 & \cdots & -1 & 1 \\ 1 & 1 & 0 & -1 & \cdots & -1 & 1 \\ 1 & 1 & 1 & 0 & \cdots & -1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 0 & 1 \end{array} \right]. \quad (2.35b)$$

Adding the second row to the first one then gives

$$\left[\begin{array}{cccccc|c} -1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & -1 & -1 & \cdots & -1 & 1 \\ 1 & 1 & 0 & -1 & \cdots & -1 & 1 \\ 1 & 1 & 1 & 0 & \cdots & -1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & 0 & 1 \end{array} \right]. \quad (2.35c)$$

We may now iteratively add the first row to all the rows below the second one. This leads to

$$\left[\begin{array}{cccccc|c} -1 & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -1 & -1 & -1 & \cdots & -1 & 1 \\ 0 & 0 & 0 & -1 & \cdots & -1 & 1 \\ 0 & 0 & 1 & 0 & \cdots & -1 & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & 1 & \cdots & 0 & 1 \end{array} \right]. \quad (2.35d)$$

Observe that the lower-right $(m-1) \times (m-1+1)$ augmented matrix is nothing but the augmented form of the equation

$$A_{(m-1)} \mathbf{k} = \mathbf{1}, \quad \mathbf{1}^\top \in (\mathbb{Z}/N\mathbb{Z})^m. \quad (2.35e)$$

We are now in a position to use our induction hypothesis.

If $m+1$ is even, then so is $m-1$. Thus, a solution $\mathbf{k}_{\text{even}}^{(m+1)}$ of Eq. (2.34) is always of the form

$$\left(\mathbf{k}_{\text{even}}^{(m+1)} \right)^\top = (x, y, \underbrace{1, -1, \dots, 1, -1}_{m-1 \text{ entries}}) \quad (2.36)$$

We deduce from the first row of the augmented matrix (2.35d) that $x = -y$ must hold. Moreover, the second row implies that $-y = 1$. We conclude that

$$\left(\mathbf{k}_{\text{even}}^{(m+1)} \right)^\top = (1, -1, 1, -1, \dots, 1, -1) \in (\mathbb{Z}/N\mathbb{Z})^{m+1}, \quad (2.37)$$

thereby validating the induction hypothesis.

If $m+1$ is odd, then so is $m-1$. The system of linear equations corresponding to the lower $m-1$ rows of the augmented matrix (2.35d) are inconsistent by the induction hypothesis. It follows that the entire system of $m+1$ linear equations is also inconsistent. Thus, there is no possible global parafermion charge operator if $m+1$ is odd. $\square$

Corollary 1 (Recovery of a global parafermion charge for odd m). Given are the odd integer $m \geq 1$ and the integer $N \geq 2$. We embed the parafermion algebra

$$\mathcal{P}_N^m = \text{span}_{\mathbb{C}} \left\{ \gamma_1^{k_1} \cdots \gamma_m^{k_m} \mid k_1, \dots, k_m = 0, 1, \dots, N-1 \right\} \quad (2.38a)$$

in the extended parafermion algebra

$$\mathcal{P}_{N,\infty}^{m+1} = \text{span}_{\mathbb{C}} \left\{ \gamma_1^{k_1} \cdots \gamma_m^{k_m} \gamma_\infty^{k_\infty} \mid k_1, \dots, k_m, k_\infty = 0, 1, \dots, N-1 \right\}. \quad (2.38b)$$

A unitary operator $U_{\text{pf},\infty} \in \mathcal{P}_{N,\infty}^{m+1}$ satisfies the property

$$U_{\text{pf},\infty} \gamma_i U_{\text{pf},\infty}^\dagger = \omega_N \gamma_i, \quad i = 1, \dots, m. \quad (2.38c)$$

if and only if

$$U_{\text{pf},\infty} := Z \gamma_\infty^\dagger, \quad Z \in \mathcal{Z}(\mathcal{P}_N^m) \quad (2.38d)$$

and

$$Z Z^\dagger = \mathbb{1}, \quad (2.38e)$$

i.e., Z is unitary.

Proof. ($\Leftarrow$) For any $i = 1, \dots, m$,

$$\gamma_\infty^\dagger \gamma_i \gamma_\infty = \omega_N \gamma_i \quad (2.39)$$

and, as any element Z from the center $\mathcal{Z}(\mathcal{P}_N^m)$ commutes with γ_i ,

$$U_{\text{pf},\infty} \gamma_i U_{\text{pf},\infty}^\dagger = Z \left(\gamma_\infty^\dagger \gamma_i \gamma_\infty \right) Z^\dagger = Z \left(\omega_N \gamma_i \right) Z^\dagger = \omega_N \gamma_i \left(Z Z^\dagger \right) = \omega_N \gamma_i. \quad (2.40)$$

($\Rightarrow$) For any $\mathbf{k} \in \mathbb{Z}_N^m$ and for any $k_\infty \in \mathbb{Z}_N$ define

$$V_{\mathbf{k},k_\infty} := W_{\mathbf{k}} \gamma_\infty^{k_\infty}, \quad W_{\mathbf{k}} := \gamma_1^{k_1} \cdots \gamma_m^{k_m}, \quad (2.41)$$

where $W_{\mathbf{k}} \in \mathcal{P}_N^m$. Recall that this set forms a basis of the parafermion algebra $\mathcal{P}_{N,\infty}^{m+1}$. By assumption, there exists N^{m+1} complex-valued coefficients $c_{\mathbf{k},k_\infty}$ such that

$$U_{\text{pf}} = \sum_{\mathbf{k} \in \mathbb{Z}_N^m} \sum_{k_\infty \in \mathbb{Z}_N} c_{\mathbf{k},k_\infty} V_{\mathbf{k},k_\infty}, \quad (2.42a)$$

$$U_{\text{pf}} U_{\text{pf}}^\dagger = \sum_{\mathbf{k}, \mathbf{l} \in \mathbb{Z}_N^m} \sum_{k_\infty, l_\infty \in \mathbb{Z}_N} c_{\mathbf{k},k_\infty} c_{\mathbf{l},l_\infty}^* V_{\mathbf{k},k_\infty} V_{\mathbf{l},l_\infty}^\dagger = \mathbb{1}, \quad (2.42b)$$

$$U_{\text{pf}} \gamma_i U_{\text{pf}}^\dagger = \sum_{\mathbf{k}, \mathbf{l} \in \mathbb{Z}_N^m} \sum_{k_\infty, l_\infty \in \mathbb{Z}_N} c_{\mathbf{k},k_\infty} c_{\mathbf{l},l_\infty}^* V_{\mathbf{k},k_\infty} \gamma_i V_{\mathbf{l},l_\infty}^\dagger \stackrel{!}{=} \omega_N \gamma_i, \quad i = 1, \dots, m. \quad (2.42c)$$

Compatibility between the condition (2.42b), namely that U_{pf} is unitary, and the condition (2.42c), namely that U_{pf} measures the parafermion charge one for the odd- m generators $\gamma_1, \dots, \gamma_m$, is met if and only if

$$\gamma_i V_{\mathbf{l},l_\infty}^\dagger = V_{\mathbf{l},l_\infty}^\dagger \omega_N \gamma_i \iff V_{\mathbf{l},l_\infty} \gamma_i V_{\mathbf{l},l_\infty}^\dagger = \omega_N \gamma_i \quad (2.43)$$

for all $i = 1, \dots, m$. Condition (2.43) becomes

$$\omega_N \gamma_i \stackrel{!}{=} V_{\mathbf{l}, l_\infty} \gamma_i V_{\mathbf{l}, l_\infty}^\dagger \quad (2.44)$$

$$= \omega_N^{-l_\infty} W_{\mathbf{l}} \gamma_i W_{\mathbf{l}}^\dagger, \quad i = 1, \dots, m, \quad (2.45)$$

which is equivalent to

$$W_{\mathbf{l}} \gamma_i W_{\mathbf{l}}^\dagger \stackrel{!}{=} \omega_N^{l_\infty+1} \gamma_i, \quad i = 1, \dots, m. \quad (2.46)$$

By **Lemma 1**, this is equivalent to the matrix equation over $(\mathbb{Z}/N\mathbb{Z})^m$

$$A_{(m)} \mathbf{l} = (1 + l_\infty) \mathbf{1}. \quad (2.47)$$

One may adapt the proof of **Proposition 2** by substituting 1 with $1 + l_\infty$ in Eq. (2.31), yielding the conclusion that the (inhomogeneous) system of linear equations (2.47) has a solution if and only if $1 + l_\infty$ vanishes as m is odd. Having fixed

$$l_\infty = -1 \quad (2.48)$$

we are left with solving the homogeneous equation

$$A_{(m)} \mathbf{l} = \mathbf{0}. \quad (2.49)$$

According to **Proposition 1** the solution of (2.49) implies that $W_{\mathbf{l}}$ must belong to the center of $\mathcal{P}_N^m$, i.e., $W_{\mathbf{l}} = Y^{q_{\mathbf{l}}}$ for some $q_{\mathbf{l}} \in \mathbb{Z}_N$. $\square$

Corollary 2 (Recovery of a unique parafermion charge for odd m). Given are the assumptions of **Corollary 1**. Up to a multiplicative phase factor,

$$Y^k \gamma_\infty^\dagger, \quad k \in \mathbb{Z}_N, \quad (2.50a)$$

is the only unitary operator that satisfies the properties

$$\left(Y^k \gamma_\infty^\dagger \right) \gamma_i \left(Y^k \gamma_\infty^\dagger \right)^\dagger = \omega_N \gamma_i, \quad i = 1, \dots, m, \quad (2.50b)$$

and

$$\left(Y^k \gamma_\infty^\dagger \right) \gamma_\infty \left(Y^k \gamma_\infty^\dagger \right)^\dagger = \omega_N^k \gamma_\infty. \quad (2.50c)$$

Chapter 3

Parafermion algebra with an even number of generators

3.1 Introduction

In this chapter, we specialize the general theory of parafermion algebras to the case of an even number of generators, where additional structure emerges. Exploiting the decomposition of the $2m$ parafermion operators into two interleaved species, we construct a natural notion of local and global parafermion charge that generalizes fermionic parity to the $\mathbb{Z}_N$ setting. We show that these charge operators are unitary, of finite order, and admit a simultaneous diagonalization, thereby endowing the Hilbert space with a $\mathbb{Z}_N$ -graded structure. This grading provides a canonical basis in which the parafermion operators admit explicit matrix representations, allowing for a concrete realization of the algebra and its action on states.

Building on this structure, we introduce and analyze charge conjugation symmetries for parafermions. We first define a canonical charge conjugation operator, characterize its algebraic properties, and show that it generalizes the familiar fermionic case. We then extend this construction to a family of generalized charge conjugations parameterized by elements of $\mathbb{Z}/N\mathbb{Z}$, determining precisely when these operators are invertible, unitary, and symmetry-preserving.

Finally, we relate these constructions to number-theoretical properties of $\mathbb{Z}/N\mathbb{Z}$, in particular the solutions of $a^2 \equiv 1 \pmod{N}$, which classify physically admissible charge conjugation symmetries. This provides a complete characterization of parafermionic charge conjugation in the even-generator case.

We begin in **Section 3.2** by restating the defining relations of the parafermion algebra in terms of two species of parafermion operators. In **Section 3.3** these are used to construct local and global parafermion charges, as well as the accompanying local charge basis of the Hilbert space. We conclude with **Section 3.4**, where we define canonical and generalized charge conjugations and derive the associated physical constraints.

3.2 Setup

Let m and N be a pair of non-vanishing positive integers. Given is the order- N parafermion algebra with an even number of generators (linear operators acting on a Hilbert space) $(\gamma_1, \dots, \gamma_{2m})$ and their adjoints $(\gamma_1^\dagger, \dots, \gamma_{2m}^\dagger)$ obeying the relations

$$\gamma_i^N = \mathbb{1}, \quad (3.1a)$$

$$\gamma_i \gamma_{i'} = \omega_N^{\text{sgn}(i'-i)} \gamma_{i'} \gamma_i, \quad (3.1b)$$

$$\gamma_i^\dagger = \gamma_i^{N-1}, \quad (3.1c)$$

where

$$\omega_N := e^{+\frac{2\pi i}{N}}, \quad \text{sgn}(i' - i) := \begin{cases} +1, & i' > i, \\ 0, & i' = i, \\ -1, & i' < i. \end{cases} \quad (3.1d)$$

In view of the fact that the number $2m$ of parafermion generators is even, we may introduce two types (flavors) of parafermion generators. For $i \in \{1, \dots, 2m\}$ with $j = \lceil \frac{i}{2} \rceil \in \{1, \dots, m\}$, we thus define

$$\gamma_i \equiv \begin{cases} \eta_j, & \text{for } i = 2j \text{ even,} \\ \xi_j, & \text{for } i = 2j - 1 \text{ odd.} \end{cases} \quad (3.1e)$$

The even parafermion algebra $\mathcal{P}_N^{2m}$ is the N^{2m} -dimensional vector space over the field of complex numbers

$$\mathcal{P}_N^{2m} := \text{span}_{\mathbb{C}} \{ \gamma_1^{r_1} \gamma_2^{r_2} \dots \gamma_m^{r_{2m}} \mid r_1, \dots, r_{2m} = 0, 1, \dots, N-1 \} \quad (3.1f)$$

equipped with the bilinear product defined by Eqs. (3.1a-3.1c).

3.3 Local and global parafermion charges

When studying fermionic systems, a key physical postulate is the existence and conservation of fermion parity. Since parafermions generalize fermions, we seek an appropriate generalization of the fermion parity. The fermion parity is a two-valued attribute. For the algebra $\mathcal{P}_N^{2m}$ (3.1) of even number of parafermions, the parafermion parity takes N distinct values that we shall call the parafermion charges.

We take advantage through Eq. (3.1e) of the existence of an even number $2m$ of generators of the parafermion algebra $\mathcal{P}_N^{2m}$.

Definition 4 (Local and global parafermion charge operators). Given is the even parafermion algebra $\mathcal{P}_N^{2m}$ (3.1). There are m *local parafermion charges*

$$U_{\text{pf},j} := \omega_N^{S_N} \gamma_{2j-1}^\dagger \gamma_{2j}^\dagger \equiv \omega_N^{\frac{N+1}{2}} \xi_j \eta_j^\dagger, \quad j = 1, \dots, m. \quad (3.2a)$$

There is the *global parafermion charge*

$$U_{\text{pf}} := \prod_{j=1}^m U_{\text{pf},j}. \quad (3.2b)$$

Example 4 (Fermion parity). When $N = 2$, the operator U_{pf} is nothing but the fermion parity in term of Majoranas [31].

Remark 6. We will show that the above definition is in fact unique (up to phase factors). Moreover, we prove that when m is odd, there exists no operator realizing global parafermion charge.

The following lemma shows that **Definition 4** is the generalization of fermion parity. In particular, we are going to construct a basis of a global Hilbert space of dimension N^m with a $\mathbb{Z}_N$ -graded structure on which the $2m$ generators of the parafermion algebra $\mathcal{P}_N^{2m}$ act.

Proposition 3 (Properties of parafermion charge and Hilbert space grading). Given are the even parafermion algebra $\mathcal{P}_N^{2m}$ (3.1), the local parafermion charges (3.2a), and the global parafermion charge (3.2b). We then have the following properties.

1. All m local parafermion charges are of order N ,

$$U_{\text{pf},j}^N = \mathbb{1}, \quad j = 1 \dots, m. \quad (3.3a)$$

2. All local parafermion charges are unitary,

$$U_{\text{pf},j}^\dagger = U_{\text{pf},j}^{-1} = U_{\text{pf},j}^{N-1}, \quad j = 1 \dots, m. \quad (3.3b)$$

3. Conjugation of the parafermion γ_i with $i = 1, \dots, 2m$ by the local parafermion charge $U_{\text{pf},j}$ with $j = 1, \dots, m$ gives

$$U_{\text{pf},j} \gamma_i U_{\text{pf},j}^\dagger = \begin{cases} \omega_N \gamma_i, & \text{if } i = 2j - 1, j, \\ \gamma_i, & \text{else.} \end{cases} \quad (3.3c)$$

4. All m local parafermion charges commute pairwise,

$$[U_{\text{pf},j}, U_{\text{pf},j'}] = 0, \quad j, j' = 1 \dots, m. \quad (3.3d)$$

5. There exists a complete and orthonormal basis of $\mathfrak{H} \cong \mathbb{C}^{N^m}$

$$\mathbb{C}^{N^m} = \text{span} \left\{ |\mathbf{k}\rangle, \mathbf{k} \in \mathbb{Z}_N^m \mid \langle \mathbf{k} | \mathbf{k}' \rangle = \delta_{\mathbf{k}, \mathbf{k}'}, \mathbf{k}, \mathbf{k}' \in \mathbb{Z}_N^m, \mathbb{1} = \sum_{\mathbf{k} \in \mathbb{Z}_N^m} |\mathbf{k}\rangle \langle \mathbf{k}| \right\} \quad (3.3e)$$

with the N^m -dimensional representation of the algebra (3.3a)-(3.3d) given by

$$U_{\text{pf},j} |\mathbf{k}\rangle = \omega_N^{k_j} |\mathbf{k}\rangle, \quad j = 1, \dots, m, \quad \mathbf{k} \in \mathbb{Z}_N^m. \quad (3.3f)$$

6. The global parafermion charge is of order N and unitary,

$$U_{\text{pf}}^\dagger = U_{\text{pf}}^{-1} = U_{\text{pf}}^{N-1}. \quad (3.3g)$$

7. The eigenvalues of U_{pf} in the basis (3.3e) and (3.3f) are given by

$$U_{\text{pf}} |\mathbf{k}\rangle = \omega_N^{k_1 + \dots + k_m} |\mathbf{k}\rangle, \quad \mathbf{k} \in \mathbb{Z}_N^m. \quad (3.3h)$$

8. The global Hilbert space $\mathfrak{H} \cong \mathbb{C}^{N^m}$ has the $\mathbb{Z}_N$ graded structure,

$$\mathfrak{H} = \bigoplus_{q=0}^N \mathfrak{H}|_{U_{\text{pf}} = \omega_N^q}. \quad (3.3i)$$

9. Conjugation of the parafermion γ_i with $i = 1, \dots, 2m$ by the global parafermion charge U_{pf} gives

$$U_{\text{pf}} \gamma_i U_{\text{pf}}^\dagger = \omega_N \gamma_i. \quad (3.3j)$$

Proof. The identities

$$1 + \dots + N - 1 = \frac{(N-1)N}{2}, \quad (3.4)$$

$$AB = qBA, \quad 0 \neq q \in \mathbb{C} \implies (AB)^N = q^{-(1+\dots+N-1)} A^N B^N = q^{-(N-1)N/2} A^N B^N, \quad (3.5)$$

$$A \rightarrow \xi_j, \quad B \rightarrow \eta_j^\dagger, \quad q \rightarrow \omega_N^{-1} \implies (AB)^N \rightarrow (\xi_j \eta_j^\dagger)^N = \omega_N^{(N-1)N/2} \xi^N \eta^{\dagger N}, \quad (3.6)$$

$$\frac{(N+1)N}{2} + \frac{(N-1)N}{2} = N^2 \quad (3.7)$$

deliver Eq. (3.3a). The first equality in Eq. (3.3b) is a consequence of the

$$U_{\text{pf},j}^\dagger U_{\text{pf},j} = (\eta_j \xi_j^\dagger) (\xi_j \eta_j^\dagger) = \eta_j (\xi_j^{N-1} \xi_j) \eta_j^{N-1} = \mathbb{1}, \quad j = 1, \dots, m. \quad (3.8)$$

The second equality in Eq. (3.3b) is a consequence of Eq. (3.3a). Equation (3.3d) is a consequence of

$$\begin{aligned} U_{\text{pf},j} U_{\text{pf},j'} &= \omega_N^{2S_N} (\xi_j \eta_j^\dagger) (\xi_{j'} \eta_{j'}^\dagger) \\ &= \omega_N^{2S_N} (\xi_j \eta_j^{N-1}) (\xi_{j'} \eta_{j'}^{N-1}) \\ &= \omega_N^{2S_N} \times \omega_N^{N-1+1} \times \omega_N^{(N-1)^2+(N-1)} (\xi_{j'} \eta_{j'}^{N-1}) (\xi_j \eta_j^{N-1}) \\ &= \omega_N^{2S_N} (\xi_{j'} \eta_{j'}^{N-1}) (\xi_j \eta_j^{N-1}) \\ &= \omega_N^{2S_N} (\xi_{j'} \eta_{j'}^\dagger) (\xi_j \eta_j^\dagger) \\ &= U_{\text{pf},j'} U_{\text{pf},j} \end{aligned} \quad (3.9)$$

for $1 \leq j < j' \leq m$. To prove Eq. (3.3c), do

$$\begin{aligned}
U_{\text{pf},j} \gamma_i U_{\text{pf},j}^\dagger &= (\gamma_{2j-1} \gamma_{2j}^\dagger) \gamma_i (\gamma_{2j} \gamma_{2j-1}^\dagger) \\
&= (\gamma_{2j-1} \gamma_{2j}^{-1}) \gamma_i (\gamma_{2j} \gamma_{2j-1}^{-1}) \\
&= \begin{cases} (\gamma_{2j-1} \gamma_{2j}^{-1}) \gamma_{2j-1} (\gamma_{2j} \gamma_{2j-1}^{-1}) & \text{if } i = 2j - 1 \\ (\gamma_{2j-1} \gamma_{2j}^{-1}) \gamma_{2j} (\gamma_{2j} \gamma_{2j-1}^{-1}) & \text{if } i = 2j \\ (\gamma_{2j-1} \gamma_{2j}^{-1}) (\gamma_{2j} \gamma_{2j-1}^{-1}) \gamma_i & \text{else} \end{cases} \\
&= \begin{cases} \omega_N \gamma_{2j-1} & \text{if } i = 2j - 1 \\ \omega_N \gamma_{2j} & \text{if } i = 2j \\ \gamma_i & \text{else} \end{cases} \tag{3.10}
\end{aligned}$$

for $j = 1, \dots, m$ and $i = 1, \dots, 2m$. Equations (3.3e) and (3.3f) follow from Eqs. (3.3a)-(3.3d):

- The fact that the local parafermion charge is of order N and unitary implies that the local parafermion charge has N (not necessarily distinct) eigenvalues that are N -th roots of unity.
- Equation (3.3c) implies that these eigenvalues are necessarily distinct.
- The fact (3.3d) that the local parafermion charges all commute pairwise implies that they can all be diagonalized simultaneously.

Properties (3.3g)-(3.3j) are inherited from Eqs. (3.3a)-(3.3f). □

Definition 5 (Local charge basis). The basis defined by Eqs. (3.3e) and (3.3f) is called the $\mathbb{Z}_N$ -graded parafermion charge basis.

We give an explicit construction of the local charge basis (3.3e) and compute the respective representation of the parafermion operators.

Lemma 4 (Representation and action of parafermion charge operators). Given are the even parafermion algebra $\mathcal{P}_N^{2m}$ (3.1), the local parafermion charges (3.2a), and the global parafermion charge (3.2b). The $\mathbb{Z}_N$ -graded parafermion charge basis (3.3e) and (3.3f) can then be realized by the (not unique) choice

$$|\mathbf{k}\rangle = \eta_1^{k_1} \dots \eta_m^{k_m} |0\rangle, \tag{3.11a}$$

where the state $|0\rangle \equiv |0\rangle$ is a normalized solution to the eigenvalue problem

$$U_{p,j} |0\rangle = |0\rangle, \quad j = 1, \dots, m. \tag{3.11b}$$

Define the not necessarily integer constant

$$S_N := \frac{N+1}{2} \in \begin{cases} \mathbb{Z}, & \text{for } N \text{ odd,} \\ \mathbb{Z} + \frac{1}{2}, & \text{for } N \text{ even.} \end{cases} \tag{3.12a}$$

The actions of the parafermionic operators on these states are

$$\eta_j |k_1, \dots, k_j, \dots, k_m\rangle := \omega_N^{-k_1 - \dots - k_{j-1}} |k_1, \dots, k_j + 1, \dots, k_m\rangle, \quad (3.12b)$$

$$\xi_j |k_1, \dots, k_j, \dots, k_m\rangle := \omega_N^{+S_N + k_j} \omega_N^{-k_1 - \dots - k_{j-1}} |k_1, \dots, k_j + 1, \dots, k_m\rangle, \quad (3.12c)$$

or, equivalently,

$$\begin{aligned} \gamma_i |k_1, \dots, k_j, \dots, k_m\rangle &:= \begin{cases} \eta_j |k_1, \dots, k_j, \dots, k_m\rangle, & \text{for } i = 2j \text{ even,} \\ \xi_j |k_1, \dots, k_j, \dots, k_m\rangle, & \text{for } i = 2j - 1 \text{ odd,} \end{cases} \\ &= \begin{cases} \omega_N^{-k_1 - \dots - k_{j-1}} |k_1, \dots, k_j + 1, \dots, k_m\rangle, & i = 2j \text{ even,} \\ \omega_N^{+S_N + k_j} \omega_N^{-k_1 - \dots - k_{j-1}} |k_1, \dots, k_j + 1, \dots, k_m\rangle, & i = 2j - 1 \text{ odd,} \end{cases} \end{aligned} \quad (3.12d)$$

We have the representations

$$\eta_j = \sum_{\mathbf{k} \in \mathbb{Z}_N^m} \omega_N^{-k_1 - \dots - k_{j-1}} |k_1, \dots, k_j + 1, \dots, k_m\rangle \langle k_1, \dots, k_j, \dots, k_m|, \quad (3.13a)$$

$$\xi_j = \sum_{\mathbf{k} \in \mathbb{Z}_N^m} \omega_N^{+S_N + k_j} \omega_N^{-k_1 - \dots - k_{j-1}} |k_1, \dots, k_j + 1, \dots, k_m\rangle \langle k_1, \dots, k_j, \dots, k_m|, \quad (3.13b)$$

the representations

$$\begin{aligned} \eta_j^\dagger &= \sum_{\mathbf{k} \in \mathbb{Z}_N^m} \omega_N^{+k_1 + \dots + k_{j-1}} |k_1, \dots, k_j, \dots, k_m\rangle \langle k_1, \dots, k_j + 1, \dots, k_m| \\ &= \sum_{\mathbf{k} \in \mathbb{Z}_N^m} \omega_N^{+k_1 + \dots + k_{j-1}} |k_1, \dots, k_j - 1, \dots, k_m\rangle \langle k_1, \dots, k_j, \dots, k_m|, \end{aligned} \quad (3.14a)$$

$$\begin{aligned} \xi_j^\dagger &= \sum_{\mathbf{k} \in \mathbb{Z}_N^m} \omega_N^{-S_N - k_j} \omega_N^{+k_1 + \dots + k_{j-1}} |k_1, \dots, k_j, \dots, k_m\rangle \langle k_1, \dots, k_j + 1, \dots, k_m| \\ &= \sum_{\mathbf{k} \in \mathbb{Z}_N^m} \omega_N^{+S_N - k_j} \omega_N^{+k_1 + \dots + k_{j-1}} |k_1, \dots, k_j - 1, \dots, k_m\rangle \langle k_1, \dots, k_j, \dots, k_m|, \end{aligned} \quad (3.14b)$$

and the representations

$$\eta_j^n = \sum_{\mathbf{k} \in \mathbb{Z}_N^m} \omega_N^{-n(k_1 + \dots + k_{j-1})} |k_1, \dots, k_j + n, \dots, k_m\rangle \langle k_1, \dots, k_j, \dots, k_m|, \quad (3.14c)$$

$$\xi_j^n = \sum_{\mathbf{k} \in \mathbb{Z}_N^m} \omega_N^{+n(S_N + k_j) + n(n-1)/2} \omega_N^{-n(k_1 + \dots + k_{j-1})} |k_1, \dots, k_j + n, \dots, k_m\rangle \langle k_1, \dots, k_j, \dots, k_m|. \quad (3.14d)$$

Proof. We do the manipulations

$$\begin{aligned}
U_{\text{pf},j} |\mathbf{k}\rangle &= U_{\text{pf},j} \left(\eta_1^{k_1} \cdots \eta_j^{k_j} \cdots \eta_m^{k_m} \right) |0\rangle \\
&= U_{\text{pf},j} \left(\eta_1^{k_1} \cdots \eta_j^{k_j} \cdots \eta_m^{k_m} \right) U_{\text{pf},j}^\dagger U_{\text{pf},j} |0\rangle \\
&= \left[\left(U_{\text{pf},j} \eta_1^{k_1} U_{\text{pf},j}^\dagger \right) \cdots \left(U_{\text{pf},j} \eta_j^{k_j} U_{\text{pf},j}^\dagger \right) \cdots \left(U_{\text{pf},j} \eta_m^{k_m} U_{\text{pf},j}^\dagger \right) \right] U_{\text{pf},j} |0\rangle \\
&= \left[\left(U_{\text{pf},j} \eta_m U_{\text{pf},j}^\dagger \right)^{k_m} \cdots \left(U_{\text{pf},j} \eta_j U_{\text{pf},j}^\dagger \right)^{k_j} \cdots \left(U_{\text{pf},j} \eta_1 U_{\text{pf},j}^\dagger \right)^{k_1} \right] U_{\text{pf},j} |0\rangle \\
&= \left[(\eta_m)^{k_1} \cdots (\eta_N \eta_j)^{k_j} \cdots (\eta_m)^{k_m} \right] U_{\text{pf},j} |0\rangle \\
&= \omega_N^{k_j} \left(\eta_1^{k_1} \cdots \eta_j^{k_j} \cdots \eta_m^{k_m} \right) U_{\text{pf},j} |0\rangle, \quad j = 1, \dots, m.
\end{aligned} \tag{3.15}$$

We made use of Eq. (3.3c) to reach the penultimate equality. With the help of Eq. (3.11b), we conclude that

$$U_{\text{pf},j} |\mathbf{k}\rangle = \omega_N^{k_j} |\mathbf{k}\rangle, \quad j = 1, \dots, m. \tag{3.16}$$

The first action (3.13b) is computed by passing η_j through the string of parafermions $(\eta_1 \cdots \eta_{j-1})$. The second action (3.12c) can be verified by first computing the action $\xi_j |0\rangle$ using the orthonormality of the basis (3.11a). Equations (3.13a) and (3.13b) follow from Eq. (3.12b) and (3.12c), respectively. The additional multiplicative phase factor

$$\omega_N^{+k_j} \tag{3.17a}$$

present on the right-hand side of Eq. (3.12c) compared to (3.12b) is needed to enforce the condition

$$\frac{\lambda_{<}}{\lambda_{>}} = \omega_N, \tag{3.17b}$$

where the $\mathbb{C}$ numbers $\lambda_{<}$ and $\lambda_{>}$ of magnitude one are defined by

$$\xi_j \eta_{j'} |k_1, \dots, k_j, \dots, k_{j'}, \dots, l_m\rangle = \lambda_{<} |k_1, \dots, k_j + 1, \dots, k_{j'} + 1, \dots, l_m\rangle, \tag{3.17c}$$

$$\eta_{j'} \xi_j |k_1, \dots, k_j, \dots, k_{j'}, \dots, l_m\rangle = \lambda_{>} |k_1, \dots, k_j + 1, \dots, k_{j'} + 1, \dots, l_m\rangle, \tag{3.17d}$$

respectively (without loss of generality $j < j'$). The additional multiplicative phase factor

$$\omega_N^{S_N} \tag{3.18a}$$

present on the right-hand side of Eq. (3.12c) compared to (3.12b) is needed to enforce the condition

$$\omega_N^{S_N} \xi_j \eta_j^\dagger |\mathbf{k}\rangle = \omega_N^{k_j} |\mathbf{k}\rangle, \tag{3.18b}$$

owing to the identity

$$S_N + (S_N - 1) = \frac{N+1}{2} + \frac{N-1}{2} = N, \tag{3.18c}$$

which implies $-S_N \equiv S_N - 1 \pmod{N}$.

We prove by induction on $n \in \mathbb{N}$ that

$$\eta_j^n |\mathbf{k}\rangle = \omega_N^{-n(k_1+\dots+k_{j-1})} |k_1, \dots, k_j + n, \dots, k_m\rangle. \quad (3.19)$$

Base case. For $n = 1$, Eq. (3.12b) gives

$$\eta_j |\mathbf{k}\rangle = \omega_N^{-(k_1+\dots+k_{j-1})} |k_1, \dots, k_j + 1, \dots, k_m\rangle, \quad (3.20)$$

so Eq. (3.19) with $n = 1$ holds.

Inductive step. Assume that Eq. (3.19) holds for n . Then

$$\begin{aligned} \eta_j^{n+1} |\mathbf{k}\rangle &= \eta_j (\eta_j^n |\mathbf{k}\rangle) \\ &= \eta_j \left(\omega_N^{-n(k_1+\dots+k_{j-1})} |k_1, \dots, k_j + n, \dots, k_m\rangle \right) \\ &= \omega_N^{-n(k_1+\dots+k_{j-1})} \omega_N^{-(k_1+\dots+k_{j-1})} |k_1, \dots, k_j + n + 1, \dots, k_m\rangle \\ &= \omega_N^{-(n+1)(k_1+\dots+k_{j-1})} |k_1, \dots, k_j + n + 1, \dots, k_m\rangle. \end{aligned} \quad (3.21)$$

This is precisely Eq. (3.19) with $n + 1$, completing the induction.

We prove by induction that

$$\xi_j^n |\mathbf{k}\rangle = \omega_N^{n(S_N+k_j)+\frac{n(n-1)}{2}} \omega_N^{-n(k_1+\dots+k_{j-1})} |k_1, \dots, k_j + n, \dots, k_m\rangle. \quad (3.22)$$

Base case. For $n = 1$, Eq. (3.12c) gives

$$\xi_j |\mathbf{k}\rangle = \omega_N^{S_N+k_j} \omega_N^{-(k_1+\dots+k_{j-1})} |k_1, \dots, k_j + 1, \dots, k_m\rangle, \quad (3.23)$$

which agrees with Eq. (3.22) for $n = 1$.

Inductive step. Assume that Eq. (3.22) holds for n . Then

$$\begin{aligned} \xi_j^{n+1} |\mathbf{k}\rangle &= \xi_j (\xi_j^n |\mathbf{k}\rangle) \\ &= \xi_j \left(\omega_N^{n(S_N+k_j)+\frac{n(n-1)}{2}} \omega_N^{-n(k_1+\dots+k_{j-1})} |k_1, \dots, k_j + n, \dots, k_m\rangle \right) \\ &= \omega_N^{n(S_N+k_j)+\frac{n(n-1)}{2}} \omega_N^{-n(k_1+\dots+k_{j-1})} \xi_j |k_1, \dots, k_j + n, \dots, k_m\rangle. \end{aligned} \quad (3.24)$$

Using the $n = 1$ case on the shifted state,

$$\xi_j |k_1, \dots, k_j + n, \dots, k_m\rangle = \omega_N^{S_N+k_j+n} \omega_N^{-(k_1+\dots+k_{j-1})} |k_1, \dots, k_j + n + 1, \dots, k_m\rangle, \quad (3.25)$$

we obtain

$$\begin{aligned} \xi_j^{n+1} |\mathbf{k}\rangle &= \omega_N^{n(S_N+k_j)+\frac{n(n-1)}{2}} \omega_N^{-n(k_1+\dots+k_{j-1})} \omega_N^{S_N+k_j+n} \omega_N^{-(k_1+\dots+k_{j-1})} |k_1, \dots, k_j + n + 1, \dots, k_m\rangle \\ &= \omega_N^{(n+1)(S_N+k_j)+\frac{(n+1)n}{2}} \omega_N^{-(n+1)(k_1+\dots+k_{j-1})} |k_1, \dots, k_j + n + 1, \dots, k_m\rangle. \end{aligned} \quad (3.26)$$

This is precisely Eq. (3.22) with $n + 1$, completing the induction. $\square$

3.4 Parafermion charge conjugation

A symmetry that is often encountered in physics is that of the charge conjugation. Charge conjugation is an involution that interchanges particles with their antiparticles. Since Majoranas operators (fermionic $N = 2$ case) are self-inverse (they square to identity), charge conjugation may act only by producing the multiplicative factors ± 1 . This is not the case for parafermions with order $N \geq 3$, for which

$$\gamma_i^\dagger = \gamma_i^{-1} \equiv \gamma_i^{N-1} \neq \gamma_i. \quad (3.27)$$

Parafermions of order N larger than two can realize the operation of charge conjugation in a way richer than when $N = 2$.

We will show how charge conjugation can be realized in systems generated by an even number of parafermions, and that for appropriate $N \geq 8$, it is possible to simultaneously realize different types of charge conjugation.

3.4.1 Canonical charge conjugation

Charge conjugation for fermions or bosons interchange creation and annihilation operators. By analogy, we seek a linear operation that maps parafermion operators and the local parafermion charges to their adjoints.

We begin by defining the canonical charge conjugation by demanding that it maps all states with given parafermion charge $\mathbf{k}$ to a state with parafermion charge $-\mathbf{k}$. Such a transformation exists for any order N .

Definition 6 (Canonical parafermion charge conjugation). Given are the even parafermion algebra $\mathcal{P}_N^{2m}$ (3.1), the local and global parafermion charges (3.2a) and (3.2b), respectively, and the $\mathbb{Z}_N$ -graded parafermion charge basis (3.3e) and (3.3f). We define the *canonical parafermion charge conjugation* operator U_c by its action

$$U_c |\mathbf{k}\rangle := |-\mathbf{k}\rangle, \quad \mathbf{k} \in \mathbb{Z}_N^m. \quad (3.28)$$

Remark 7. As we will see, the operator (3.28) always realizes charge conjugation. We will show that whenever N is an odd prime power, U_c is the *only* possible non-trivial charge conjugation. This justifies the use of the term “canonical”.

The following lemma justifies **Definition 6**.

Lemma 5 (Transformations under canonical parafermion charge conjugation). Given are the even parafermion algebra $\mathcal{P}_N^{2m}$ (3.1), the local and global parafermion charges (3.2a) and (3.2b), respectively, the $\mathbb{Z}_N$ -graded parafermion charge basis (3.3e) and (3.3f), and the

parafermion charge conjugation operator (3.28). We then have the identities

$$U_c = \sum_{\mathbf{k} \in \mathbb{Z}_N^m} |-\mathbf{k}\rangle \langle +\mathbf{k}|, \quad (3.29a)$$

$$U_c^2 = \mathbb{1}, \quad (3.29b)$$

$$U_c^\dagger = U_c, \quad (3.29c)$$

$$U_c \gamma_i U_c^\dagger = \gamma_i^\dagger, \quad i = 1, \dots, 2m, \quad (3.29d)$$

$$U_c U_{\text{pf},j} U_c^\dagger = U_{\text{pf},j}^\dagger, \quad j = 1, \dots, m, \quad (3.29e)$$

$$U_c U_{\text{pf}} U_c^\dagger = U_{\text{pf}}^\dagger. \quad (3.29f)$$

Remark 8 (Triviality of U_c when $N = 2$). For $N = 2$, all $2m$ parafermion operators and all m local parafermion charges are Hermitian operators. Hence, any parafermion operator or any local parafermion charge is invariant under conjugation by U_c when $N = 2$. This is to say that U_c is not realizing the charge conjugation by which fermion creation and annihilation operators are exchanged. Such an alternative transformation is achieved by imposing the linear transformation rules $\xi_j \mapsto +\xi_j$ and $\eta_j \mapsto -\eta_j$ under which $U_{\text{pf},j} \mapsto -U_{\text{pf},j}$ if we have declared that $c_j := \xi_j + i\eta_j$ is an annihilation fermion operator for $j = 1, \dots, m$ when $N = 2$.

Proof. Equation (3.28) is equivalent to the N^m -dimensional representation

$$U_c = \sum_{\mathbf{l} \in \mathbb{Z}_N^m} \sum_{\mathbf{l}' \in \mathbb{Z}_N^m} |\mathbf{l}'\rangle \langle \mathbf{l}'| U_c |\mathbf{l}\rangle \langle \mathbf{l}| = \sum_{\mathbf{l} \in \mathbb{Z}_N^m} \sum_{\mathbf{l}' \in \mathbb{Z}_N^m} \delta_{\mathbf{l}', -\mathbf{l}} |\mathbf{l}'\rangle \langle +\mathbf{l}| = \sum_{\mathbf{l} \in \mathbb{Z}_N^m} |-\mathbf{l}\rangle \langle +\mathbf{l}| \quad (3.30)$$

owing to the completeness and orthonormality of the $\mathbb{Z}_N$ -graded parafermion charge basis (3.3e) and (3.3f). Insertion of Eq. (3.29a) into the left-hand side of Eq. (3.29b) gives

$$U_c^2 = \sum_{\mathbf{l} \in \mathbb{Z}_N^m} \sum_{\mathbf{l}' \in \mathbb{Z}_N^m} |-\mathbf{l}\rangle \langle +\mathbf{l}| -\mathbf{l}'\rangle \langle +\mathbf{l}'| = \sum_{\mathbf{l} \in \mathbb{Z}_N^m} \sum_{\mathbf{l}' \in \mathbb{Z}_N^m} |-\mathbf{l}\rangle \delta_{\mathbf{l}, -\mathbf{l}'} \langle +\mathbf{l}'| = \sum_{\mathbf{l} \in \mathbb{Z}_N^m} |\mathbf{l}\rangle \langle \mathbf{l}| = \mathbb{1}. \quad (3.31)$$

In other words, U_c is of order 2, i.e., U_c is its own inverse. Taking the adjoint of Eq. (3.29a) gives

$$U_c^\dagger = \sum_{\mathbf{k} \in \mathbb{Z}_N^m} |+\mathbf{k}\rangle \langle -\mathbf{k}| = \sum_{\mathbf{l} \in \mathbb{Z}_N^m} |-\mathbf{l}\rangle \langle +\mathbf{l}| = U_c \quad (3.32)$$

upon the relabeling $\mathbf{k} = -\mathbf{l}$. Hence,

$$U_c = U_c^\dagger = U_c^{-1} \quad (3.33)$$

is simultaneously Hermitian and unitary. In terms of the Cartesian basis $\{\mathbf{e}_j\}_{j=1}^m$ of $\mathbb{Z}^m$, we may rewrite Eqs. (3.13a) and (3.13b) as

$$\eta_j = \sum_{\mathbf{k} \in \mathbb{Z}_N^m} \omega_N^{-k_1 - \dots - k_{j-1}} |\mathbf{k} + \mathbf{e}_j\rangle \langle \mathbf{k}|, \quad (3.34)$$

$$\xi_j = \sum_{\mathbf{k} \in \mathbb{Z}_N^m} \omega_N^{+S_N + k_j} \omega_N^{-k_1 - \dots - k_{j-1}} |\mathbf{k} + \mathbf{e}_j\rangle \langle \mathbf{k}|. \quad (3.35)$$

By taking advantage of the fact that the map $\mathbf{k} \mapsto -\mathbf{k}$ is a bijection on $\mathbb{Z}_N^m$, we may then do the manipulations

$$\begin{aligned}
U_c \eta_j U_c^\dagger &= \sum_{\mathbf{l} \in \mathbb{Z}_N^m} \sum_{\mathbf{m} \in \mathbb{Z}_N^m} \sum_{\mathbf{k} \in \mathbb{Z}_N^m} \omega_N^{-k_1 \dots - k_{j-1}} |-\mathbf{l}\rangle \langle +\mathbf{l} | \mathbf{k} + \mathbf{e}_j \rangle \langle \mathbf{k} | + \mathbf{m} \rangle \langle -\mathbf{m} | \\
&= \sum_{\mathbf{k} \in \mathbb{Z}_N^m} \omega_N^{-k_1 \dots - k_{j-1}} |-\mathbf{k} - \mathbf{e}_j\rangle \langle -\mathbf{k} | \\
&= \sum_{\mathbf{l} \in \mathbb{Z}_N^m} \omega_N^{+l_1 + \dots + l_{j-1}} |\mathbf{l} - \mathbf{e}_j\rangle \langle \mathbf{l} | \\
&= \eta_j^\dagger
\end{aligned} \tag{3.36}$$

and

$$\begin{aligned}
U_c \xi_j U_c^\dagger &= \sum_{\mathbf{l} \in \mathbb{Z}_N^m} \sum_{\mathbf{m} \in \mathbb{Z}_N^m} \sum_{\mathbf{k} \in \mathbb{Z}_N^m} \omega_N^{+S_N + k_j} \omega_N^{-k_1 \dots - k_{j-1}} |-\mathbf{l}\rangle \langle +\mathbf{l} | \mathbf{k} + \mathbf{e}_j \rangle \langle \mathbf{k} | + \mathbf{m} \rangle \langle -\mathbf{m} | \\
&= \sum_{\mathbf{k} \in \mathbb{Z}_N^m} \omega_N^{+S_N + k_j} \omega_N^{-k_1 \dots - k_{j-1}} |-\mathbf{k} - \mathbf{e}_j\rangle \langle -\mathbf{k} | \\
&= \sum_{\mathbf{l} \in \mathbb{Z}_N^m} \omega_N^{+S_N - l_j} \omega_N^{+l_1 + \dots + l_{j-1}} |\mathbf{l} - \mathbf{e}_j\rangle \langle \mathbf{l} | \\
&= \xi_j^\dagger
\end{aligned} \tag{3.37}$$

for $j = 1, \dots, m$. To reach the last equality, we made use of the identity

$$1 - S_N \equiv \frac{1 - N}{2} = \frac{1 - N}{2} + N - N = \frac{1 + N}{2} - N = S_N \pmod{N} \tag{3.38}$$

when taking the adjoint of Eq. (3.35). Finally, for any $j = 1, \dots, m$ we do the manipulations

$$\begin{aligned}
U_c U_{\text{pf},j} U_c^\dagger &= U_c \left(\omega_N^{S_N} \xi_j \eta_j^\dagger \right) U_c^\dagger \\
&= \omega_N^{S_N} \left(U_c \xi_j U_c^\dagger \right) \left(U_c \eta_j U_c^\dagger \right)^\dagger \\
&= \omega_N^{S_N} \xi_j^\dagger \eta_j \\
&= \omega_N^{S_N - 1} \eta_j \xi_j^\dagger \\
&= \omega_N^{-S_N} \eta_j \xi_j^\dagger \\
&= U_{\text{pf},j}^\dagger,
\end{aligned} \tag{3.39}$$

to prove Eq. (3.29e). Finally, Eq. (3.29e) follows from Eq. (3.29e), unitarity of U_c , and the fact that the $U_{\text{pf},j}$ commute pairwise. $\square$

3.4.2 Generalized charge conjugation

We move to the question of realizing more general notions of charge conjugation. Definition (3.28) of the canonical charge conjugation U_c may be interpreted as stating that every local

charge characterizing a state is multiplied by -1 (modulo N). Is it possible to define an analogous operator, where instead of multiplying the local charges by -1 , we multiply them by a generic integer $a \in \mathbb{Z}_N$ modulo N ?

Before defining such an operator, we restate some key facts on arithmetics modulo integers.

Fact 1 (Integer modulo N arithmetics). Let N be a non-vanishing integer.

1. The set $\{0, 1, \dots, N-1\}$ with addition modulo N defines the group $\mathbb{Z}_N$, which is isomorphic to the cyclic group of order N . The additive (i.e., group) inverse of $a \in \mathbb{Z}_N$ is given by

$$-a \equiv N - a. \quad (3.40a)$$

2. The set $\{0, 1, \dots, N-1\}$ with addition and multiplication modulo N defines the ring $\mathbb{Z}/N\mathbb{Z}$. If there exists for $a \in \mathbb{Z}/N\mathbb{Z}$ an integer $b \in \mathbb{Z}/N\mathbb{Z}$ such that

$$ab = 1 \pmod{N}, \quad (3.40b)$$

we say that a is a unit of the ring $\mathbb{Z}/N\mathbb{Z}$. In that case, the multiplicative inverse is unique and we write a^{-1} .

3. $a \in \mathbb{Z}/N\mathbb{Z}$ is a unit if and only if $\gcd(a, N) = 1$, i.e., a and N are coprime.
4. If $a \in \mathbb{Z}/N\mathbb{Z}$ is not a unit, then there exists $0 \neq b \in \mathbb{Z}/N\mathbb{Z}$ such that $ab = 0 \pmod{N}$ (i.e., a is a zero-divisor). In that case, the map $x \mapsto ax$ for $x \in \mathbb{Z}/N\mathbb{Z}$ is not injective.
5. The multiplication map

$$\begin{aligned} m_a : \mathbb{Z}/N\mathbb{Z} &\longrightarrow \mathbb{Z}/N\mathbb{Z} \\ x &\longmapsto ax \end{aligned} \quad (3.40c)$$

is a bijection if and only if a is a unit.

The multiplication of a vector $\mathbf{k} \in \mathbb{Z}_N^m$ of integers modulo N with a scalar $a \in \mathbb{Z}/N\mathbb{Z}$ is to be understood component-wise. More details on the ring $\mathbb{Z}/N\mathbb{Z}$ of integers modulo N can be found in **Appendix A**.

We are now in a position to define a generalized notion of charge conjugation through the multiplication of integers modulo N .

Definition 7 (Generalized parafermion charge conjugation). Given are the even parafermion algebra $\mathcal{P}_N^{2m}$ (3.1), the local and global parafermion charges (3.2a) and (3.2b), respectively, and the $\mathbb{Z}_N$ -graded parafermion charge basis (3.3e) and (3.3f). For any

$$a \in \mathbb{Z}/N\mathbb{Z} \quad (3.41a)$$

we define the *generalized charge conjugation* U_a by its action

$$U_a |\mathbf{k}\rangle := |a\mathbf{k}\rangle, \quad \mathbf{k} \in \mathbb{Z}_N^m, \quad a\mathbf{k} \in \mathbb{Z}_N^m. \quad (3.41b)$$

Example 5. The choice $a = -1$ is nothing but the parafermion charge conjugation operator (3.28),

$$U_c \equiv U_{-1}. \quad (3.42)$$

We first establish the representation, unitarity, and multiplicativity of the generalized charge conjugation U_a .

Lemma 6 (Properties of generalized parafermion charge conjugation). Given are the even parafermion algebra $\mathcal{P}_N^{2m}$ (3.1), the local and global parafermion charges (3.2a) and (3.2b), respectively, the $\mathbb{Z}_N$ -graded parafermion charge basis (3.3e) and (3.3f), and the integers

$$0 \neq a, b, c \in \mathbb{Z}/N\mathbb{Z} \quad (3.43a)$$

with

$$a \mathbf{k}, b \mathbf{k}, a \mathbf{k} \in \mathbb{Z}_N^m \quad (3.43b)$$

for any $\mathbf{k} \in \mathbb{Z}_N^m$. We then have the identities

$$U_a = \sum_{\mathbf{k} \in \mathbb{Z}_N^m} |a \mathbf{k}\rangle \langle \mathbf{k}|, \quad (3.44a)$$

$$U_a^\dagger = \sum_{\mathbf{k} \in \mathbb{Z}_N^m} |\mathbf{k}\rangle \langle a \mathbf{k}|, \quad (3.44b)$$

$$U_a U_a^\dagger = \sum_{\mathbf{k} \in \mathbb{Z}_N^m} |a \mathbf{k}\rangle \langle a \mathbf{k}|, \quad (3.44c)$$

$$U_a^\dagger U_a = \sum_{\mathbf{k}, \mathbf{l} \in \mathbb{Z}_N^m} \delta_{a \mathbf{k}, a \mathbf{l}} |\mathbf{k}\rangle \langle \mathbf{l}|, \quad (3.44d)$$

$$U_a U_b = U_{ab}, \quad (3.44e)$$

$$(U_a U_b) U_c = U_a (U_b U_c). \quad (3.44f)$$

Proof. Equation (3.44a) is a consequence of the completeness and orthonormality of the $\mathbb{Z}_N$ -graded parafermion charge basis. Equation (3.44b) is the definition of the adjoint. Equations (3.44c) and (3.44d) are consequences of the orthonormality of the $\mathbb{Z}_N$ -graded parafermion charge basis. Equation (3.44e) is a consequence of

$$\begin{aligned} U_a U_b &= \sum_{\mathbf{k} \in \mathbb{Z}_N^m} \sum_{\mathbf{l} \in \mathbb{Z}_N^m} |a \mathbf{k}\rangle \langle \mathbf{k}| b \mathbf{l}\rangle \langle \mathbf{l}| \\ &= \sum_{\mathbf{k} \in \mathbb{Z}_N^m} |(ab) \mathbf{k}\rangle \langle \mathbf{k}| \\ &= U_{ab}, \quad a, b, ab \in \mathbb{Z}/N\mathbb{Z}. \end{aligned} \quad (3.45)$$

Equation (3.44f) is a consequence of the associativity of the multiplication of integers modulo N . $\square$

Equations (3.44c) and (3.44d) do not imply that U_a is unitary whenever the map $\mathbf{k} \mapsto a \mathbf{k}$ is not a bijection on $\mathbb{Z}_N^m$. The next lemma characterizes the invertibility of U_a and ties it to its unitarity.

Lemma 7 (Unitarity of generalized parafermion charge conjugation). Given are the even parafermion algebra $\mathcal{P}_N^{2m}$ (3.1), the local and global parafermion charges (3.2a) and (3.2b), respectively, the $\mathbb{Z}_N$ -graded parafermion charge basis (3.3e) and (3.3f), and the integer

$$0 \neq a \in \mathbb{Z}/N\mathbb{Z} \quad (3.46a)$$

with

$$a \mathbf{k} \in \mathbb{Z}_N^m \quad (3.46b)$$

for any $\mathbf{k} \in \mathbb{Z}_N^m$.

1. The operator U_a defined by Eq. (3.41b) is invertible if and only if a is a unit of $\mathbb{Z}/N\mathbb{Z}$.
2. If a is a unit, then U_a is unitary with

$$U_a^\dagger = U_a^{-1} = U_{a^{-1}}. \quad (3.47)$$

3. If a is a unit, then U_a 's action (through conjugation) on the parafermions is

$$U_a \eta_j U_a^\dagger = \sum_{\mathbf{l} \in \mathbb{Z}_N^m} \omega_N^{-a^{-1}(l_1 + \dots + l_{j-1})} |\mathbf{l} + a \mathbf{e}_j\rangle \langle \mathbf{l}|, \quad (3.48)$$

$$U_a \xi_j U_a^\dagger = \sum_{\mathbf{l} \in \mathbb{Z}_N^m} \omega_N^{+S_N + a^{-1}l_j} \omega_N^{-a^{-1}(l_1 + \dots + l_{j-1})} |\mathbf{l} + a \mathbf{e}_j\rangle \langle \mathbf{l}|, \quad (3.49)$$

for $j = 1, \dots, m$. Here, $\mathbf{e}_j$ is the j -th standard basis vector of $\mathbb{Z}_N^m$.

Proof. We start with claim 1.

($\Leftarrow$) Suppose a is a unit with multiplicative inverse a^{-1} . By Eq. (3.44e)

$$U_a U_{a^{-1}} = U_{a a^{-1}} = U_1 \equiv \mathbb{1}. \quad (3.50a)$$

Thus $U_a^{-1} = U_{a^{-1}}$, which also yields the second statement of the lemma after use Eq. (3.44c).

($\Rightarrow$) Suppose a is not a unit, which implies that a is a zero-divisor. Define

$$\mathbf{0} := (0, \dots, 0) \in \mathbb{Z}_N^m \quad (3.50b)$$

and

$$\mathbf{b} := (b, \dots, b) \in \mathbb{Z}_N^m, \quad (3.50c)$$

where

$$0 \neq b, \quad a b = 0 \pmod{N}. \quad (3.50d)$$

By assumption

$$0 \neq \mathbf{b}, \quad |0\rangle \neq |\mathbf{b}\rangle, \quad (3.50e)$$

but $|0\rangle$ and $|\mathbf{b}\rangle$ share the same image,

$$U_a |0\rangle = U_a |\mathbf{b}\rangle = |0\rangle. \quad (3.50f)$$

Thus, U_a is non-injective, and hence not invertible.

We prove claim 2. According to Eqs. (3.44c) and (3.44d), we only need to prove that

$$U_a U_a^\dagger = \sum_{\mathbf{k} \in \mathbb{Z}_N^m} |a \mathbf{k}\rangle \langle a \mathbf{k}| = 1. \quad (3.50g)$$

That this is so follows from the fact that the change of variable

$$\mathbf{k} = a^{-1} \mathbf{l} \quad (3.50h)$$

is a bijection on $\mathbb{Z}_N^m$, whence we may perform a substitution in the sum.

We prove claim 3. To prove Eqs. (3.48) and (3.49) for $j = 1, \dots, m$, we do the manipulations

$$\begin{aligned} \tilde{\eta}_j &\equiv U_a \eta_j U_a^\dagger \\ &= \sum_{\mathbf{l} \in \mathbb{Z}_N^m} \sum_{\mathbf{m} \in \mathbb{Z}_N^m} \sum_{\mathbf{k} \in \mathbb{Z}_N^m} \omega_N^{-k_1 - \dots - k_{j-1}} |a \mathbf{l}\rangle \langle \mathbf{l} | \mathbf{k} + \mathbf{e}_j \rangle \langle \mathbf{k} | \mathbf{m} \rangle \langle a \mathbf{m}| \\ &= \sum_{\mathbf{k} \in \mathbb{Z}_N^m} \omega_N^{-k_1 - \dots - k_{j-1}} |a (\mathbf{k} + \mathbf{e}_j)\rangle \langle a \mathbf{k}| \\ &= \sum_{\mathbf{l} \in \mathbb{Z}_N^m} \omega_N^{-a^{-1}(l_1 + \dots + l_{j-1})} |\mathbf{l} + a \mathbf{e}_j\rangle \langle \mathbf{l}| \end{aligned} \quad (3.50i)$$

and

$$\begin{aligned} \tilde{\xi}_j &\equiv U_a \xi_j U_a^\dagger \\ &= \sum_{\mathbf{l} \in \mathbb{Z}_N^m} \sum_{\mathbf{m} \in \mathbb{Z}_N^m} \sum_{\mathbf{k} \in \mathbb{Z}_N^m} \omega_N^{+S_N + k_j} \omega_N^{-k_1 - \dots - k_{j-1}} |a \mathbf{l}\rangle \langle \mathbf{l} | \mathbf{k} + \mathbf{e}_j \rangle \langle \mathbf{k} | \mathbf{m} \rangle \langle a \mathbf{m}| \\ &= \sum_{\mathbf{k} \in \mathbb{Z}_N^m} \omega_N^{+S_N + k_j} \omega_N^{-k_1 - \dots - k_{j-1}} |a (\mathbf{k} + \mathbf{e}_j)\rangle \langle a \mathbf{k}| \\ &= \sum_{\mathbf{l} \in \mathbb{Z}_N^m} \omega_N^{+S_N + a^{-1} l_j} \omega_N^{-a^{-1}(l_1 + \dots + l_{j-1})} |\mathbf{l} + a \mathbf{e}_j\rangle \langle \mathbf{l}|, \end{aligned} \quad (3.50j)$$

for $j = 1, \dots, m$, respectively.

□

In quantum mechanics, symmetries for bosons or fermions are realized by unitary operators that leave the boson or fermion algebra invariant under conjugation. To interpret U_a as a parafermion symmetry, we thus also demand that U_a leaves the parafermion algebra invariant under conjugation.

Proposition 4 (Algebra-preserving generalized parafermion charge conjugation). Given are the even parafermion algebra $\mathcal{P}_N^{2m}$ (3.1), the local and global parafermion charges (3.2a) and (3.2b), respectively, the $\mathbb{Z}_N$ -graded parafermion charge basis (3.3e) and (3.3f), and the integer

$$a \in \mathbb{Z}/N\mathbb{Z}, \quad \gcd(a, N) = 1, \quad (3.51a)$$

so that the operator U_a defined by Eq. (3.41b) is unitary. The operator U_a leaves the $\mathbb{Z}_N$ -graded parafermion algebra $\mathcal{P}_N^{2m}$ (3.1) invariant under conjugation if and only if

$$a^2 = 1 \in \mathbb{Z}/N\mathbb{Z}, \quad (3.51b)$$

i.e., a is its own multiplicative inverse in $\mathbb{Z}/N\mathbb{Z}$. This is equivalent to the congruence problem

$$a^2 \equiv 1 \pmod{N}. \quad (3.51c)$$

Proof. According to Eqs. (3.48) and (3.49), conjugation by U_a of γ_i with $i = 1, \dots, 2m$ is closed in the $\mathbb{Z}_N$ -graded parafermion algebra (3.1) if and only if $a^{-1} = a$, in which case

$$\tilde{\gamma}_i \equiv U_a \gamma_i U_a^\dagger = \omega_N^{\Theta_{i,N,a}} \gamma_i^a, \quad i = 1, \dots, 2m, \quad (3.52a)$$

where

$$\Theta_{i,N,a} = \begin{cases} 0, & \text{if } i = 2j, \\ (1-a)S_N - \frac{a(a-1)}{2}, & \text{if } i = 2j-1. \end{cases} \quad (3.52b)$$

It follows that

$$\tilde{\gamma}_i^N = \left(\omega_N^{\Theta_{i,N,a}} \gamma_i^a \right)^N = \left(\omega_N^{Na\Theta_{i,N,a}} \gamma_i^N \right)^a = \mathbb{1}, \quad (3.53a)$$

$$\tilde{\gamma}_i \tilde{\gamma}_{i'} = \gamma_i^a \gamma_{i'}^a = \omega_N^{\Theta_{i,N,a}} \omega_N^{\Theta_{i',N,a}} \omega_N^{a^2} \gamma_{i'}^a \gamma_i^a = \omega_N \left(\omega_N^{\Theta_{i',N,a}} \gamma_{i'}^a \right) \left(\omega_N^{\Theta_{i,N,a}} \gamma_i^a \right) = \omega_N \tilde{\gamma}_{i'} \tilde{\gamma}_i, \quad (3.53b)$$

$$\tilde{\gamma}_i^\dagger = \left(\omega_N^{\Theta_{i,N,a}} \gamma_i^a \right)^\dagger = \omega_N^{-\Theta_{i,N,a}} \left(\gamma_i^\dagger \right)^a = \omega_N^{-\Theta_{i,N,a}} (\gamma_i^{N-1})^a = \omega_N^{-\Theta_{i,N,a}} (\gamma_i^a)^{N-1} = \tilde{\gamma}_i^{N-1}, \quad (3.53c)$$

for $1 \leq i < i' \leq 2m$. To reach the last equality of Eq. (3.53c), we made use of

$$\omega_N^{-\Theta_{i,N,a}} (\gamma_i^a)^{N-1} = \omega_N^{-\Theta_{i,N,a} - (N-1)\Theta_{i,N,a}} \tilde{\gamma}_i^{N-1} = \omega_N^{-N\Theta_{i,N,a}} \tilde{\gamma}_i^{N-1} = \tilde{\gamma}_i^{N-1}. \quad (3.54)$$

□

We give an example of parafermion algebra $\mathcal{P}_N^m$ that can accommodate a multiplicity of parafermion charge conjugations.

Example 6 (Three charge conjugations for order-8 parafermions). Consider parafermions of order $N = 8$. One computes

$$\begin{aligned} 0^2 &= 0 \equiv 0 \pmod{8}, \\ 1^2 &= 1 \equiv 1 \pmod{8}, \\ 2^2 &= 4 \equiv 4 \pmod{8}, \\ 3^2 &= 9 \equiv 1 \pmod{8}, \\ 4^2 &= 16 \equiv 0 \pmod{8}, \\ 5^2 &= 25 \equiv 1 \pmod{8}, \\ 6^2 &= 36 \equiv 0 \pmod{8}, \\ 7^2 &= 49 \equiv 1 \pmod{8}. \end{aligned} \quad (3.55)$$

Thus, the multiplicatively self-inverse elements of $\mathbb{Z}/8\mathbb{Z}$ are $\{1, 3, 5, 7\}$. By **Proposition 4**, the physically valid (non-trivial) operators realizing charge conjugation of order-8 parafermions are

$$U_3, U_5, U_7. \quad (3.56)$$

Since $3 \cdot 5 = 15 \equiv 7 \pmod{8}$, we deduce that the above operators realize the group structure of the Klein group, namely

$$\langle U_3, U_5, U_7 \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2. \quad (3.57)$$

We close with the solutions to $a^2 \equiv 1 \pmod{N}$ for generic integers $N \geq 2$.

Fact 2 (Number of multiplicative self-inverses modulo N). Let $N \geq 2$ be an integer. Let $a \in \mathbb{Z}/N\mathbb{Z}$ be a solution to the congruence equation

$$a^2 \equiv 1 \pmod{N}. \quad (3.58a)$$

1. If N is an odd prime power ($N = p^r$ for $p \geq 3$ a prime number, $r \geq 1$ an integer), then

$$a \equiv \pm 1. \quad (3.58b)$$

2. If N is a power of 2, then

$$a \equiv \begin{cases} 1, & \text{if } N = 2, \\ \pm 1, & \text{if } N = 4, \\ \pm 1, \pm(1 + 2^{r-1}), & \text{if } N = 2^r, r \geq 3. \end{cases} \quad (3.58c)$$

3. For general N , the solutions a are deduced from the prime power decomposition of N via the Chinese remainder theorem (see **Theorem 3**), Eq. (3.58b), and Eq. (3.58c).

More details can be found in **Appendix A**. In particular, the complete solution to the congruence problem $a^2 \equiv 1 \pmod{N}$ for general $N \geq 1$ is presented in **Example 14**.

Chapter 4

ITPs and SPTs for parafermions

4.1 Introduction

Chapter 4 develops the framework for classifying invertible and symmetry-protected topological phases (ITPs and SPTs) for one-dimensional parafermion lattice models, with the primary goal of motivating the assumptions underlying our approach.

Starting from a concrete lattice realization, we use in **Section 4.2** the structure of the parafermion algebra and its symmetries to justify key requirements such as on-site factorizability and the presence of a conserved global $\mathbb{Z}_N$ -valued parafermion charge. These considerations naturally lead to a description of the protecting symmetries as a general group extension of the parafermion charge symmetry. Unlike Ref. [31] that assumes that the fermion parity generates a central subgroup of the group extension, we ask only that the global parafermion charge generates a normal subgroup of the group extension.

We introduce in **Section 4.3** families of local Hamiltonians and motivate assumptions on locality, symmetry preservation, and the spectral gap by their role in ensuring stable phases with well-defined boundary behavior, in particular the emergence of parafermionic boundary zero modes.

Finally, we analyze in **Section 4.4** symmetry actions on bulk and boundary operators, motivating why bulk symmetries act linearly while boundary symmetries are realized projectively, subject to compatibility constraints. In particular, we include in our projective factors non-scalar operators from the algebra's center.

Together, these arguments provide a coherent justification for the assumptions used in the classification of parafermionic SPT phases.

4.2 One-dimensional chain of parafermions with their symmetries

We begin by describing our one-dimensional base space and accompanying degrees of freedom.

Let $L \gg 1$ and $N \geq 2$ be two non-vanishing integers. We consider an ordered one-dimensional lattice

$$\Lambda = \{1, 2, \dots, 2L - 1, 2L\} \tag{4.1a}$$

made of an even number $2L$ of sites. Each site $i \in \Lambda$ is assigned the order- N parafermion operator $\tilde{\gamma}_i$ and its adjoint $\tilde{\gamma}_i^\dagger$ and the pair of ordered sets

$$(\tilde{\gamma}_1, \dots, \tilde{\gamma}_{2L}) \quad (4.1b)$$

and

$$(\tilde{\gamma}_1^\dagger, \dots, \tilde{\gamma}_{2L}^\dagger) \quad (4.1c)$$

obey the relations defined in Eqs. (3.1a)-(3.1c). We may thus assign to the lattice Λ the parafermion algebra

$$\tilde{\mathcal{P}}_N^{2L} := \text{span}_{\mathbb{C}} \left\{ \tilde{\gamma}_1^{k_1} \tilde{\gamma}_2^{k_2} \dots \tilde{\gamma}_{2L}^{k_{2L}} \mid k_1, \dots, k_{2L} = 0, 1, \dots, N-1 \right\} \quad (4.1d)$$

generated by $2L$ order- N parafermions all sharing the common Hilbert space $\tilde{\mathfrak{H}}$ as domain of definition, as studied in **Chapter 3**. We have placed the symbol $\sim$ on parafermion operators and their algebra to emphasize that the chain is treated in its entirety. Upon considering a single boundary of an open chain, we shall drop the symbol $\sim$ when referring to the parafermion operators they host and the algebra they generate.

As there are an even number of sites in the one-dimensional lattice (4.1a) and Λ is the disjoint union of the even and odd sites,

$$\Lambda = \Lambda_o \sqcup \Lambda_e, \quad \Lambda_o = \{1, 3, \dots, 2L-1\}, \quad \Lambda_e = \{2, 4, \dots, 2L\}, \quad (4.2a)$$

we may group pairs of neighboring sites into L unit cells labeled by $j = 1, \dots, L$. As shown in **Chapter 3**, each unit cell $j = 1, \dots, L$ carries the local parafermion charge operator

$$\tilde{U}_{\text{pf},j} = \omega_N^{S_N} \tilde{\gamma}_{2j-1} \tilde{\gamma}_{2j}^\dagger \quad (4.2b)$$

and we may assign to the lattice Λ the global parafermion charge

$$\tilde{U}_{\text{pf}} = \prod_{j=1}^L \omega_N^{S_N} \tilde{\gamma}_{2j-1} \tilde{\gamma}_{2j}^\dagger \quad (4.2c)$$

together with the $\mathbb{Z}_N$ -graded Hilbert space

$$\begin{aligned} \tilde{\mathfrak{H}}_\Lambda &= \text{span}_{\mathbb{C}} \left\{ |\mathbf{k}\rangle \mid \tilde{U}_{\text{pf},j} |\mathbf{k}\rangle = \omega_N^{k_j} |\mathbf{k}\rangle, \mathbf{k} \in \mathbb{Z}_N^m \right\} \\ &= \bigoplus_{q=0}^{N-1} \text{span}_{\mathbb{C}} \left\{ |\mathbf{k}\rangle \mid \tilde{U}_{\text{pf}} |\mathbf{k}\rangle = \omega_N^q |\mathbf{k}\rangle, \mathbf{k} \in \mathbb{Z}_N^m \right\} \end{aligned} \quad (4.2d)$$

as the smallest Hilbert space on which the parafermion algebra $\tilde{\mathcal{P}}_N^{2L}$ is realized linearly.¹

The global parafermion charge operator (4.2c) is an example of an operator that is on-site factorizable in a homogeneous way.

¹Whereas the dimensionality of the Hilbert space $\tilde{\mathfrak{H}}_\Lambda$ is less or equal to that of $\tilde{\mathfrak{H}}$ on which the parafermion operators are defined in Eq. (4.1d), the converse is not true.

Definition 8 (Operator (homogeneous) on-site factorizability). Any operator W drawn from the parafermion algebra $\tilde{\mathcal{P}}_N^{2L}$ is on-site factorizable if it can be written as

$$W = \prod_{j=1}^L \tilde{W}_j = \tilde{W}_1 \cdots \tilde{W}_L, \quad (4.3a)$$

whereby the order of the factors $\tilde{W}_j$ is inconsequential as we demand that each pair of factors commute,

$$[\tilde{W}_j, \tilde{W}_l] = 0, \quad i, j = 1, \dots, L. \quad (4.3b)$$

Here,

$$\tilde{W}_j \in \text{span}_{\mathbb{C}} \left\{ \tilde{\gamma}_{2j-1}^{k_{2j-1}} \tilde{\gamma}_{2j}^{k_{2j}} \mid k_{2j-1}, k_{2j} = 0, 1, \dots, N-1 \right\}. \quad (4.3c)$$

If the functional form of $\tilde{W}_j$ is independent of $j = 1, \dots, L$, we say that the on-site factorizability is homogeneous.

Example 7 (Translation is not on-site factorizable). If we identify the chain Λ with a ring,

$$\Lambda \cong \mathbb{Z}_{2L}, \quad (4.4)$$

then the linear map defined by

$$\tilde{\gamma}_i \mapsto \tilde{\gamma}_{i+1}, \quad i \in \mathbb{Z}_{2L}, \quad (4.5)$$

is not on-site factorizable.

Remark 9. The set of integers $\mathbb{Z}$ is ordered and this ordering is compatible with the addition. The set $\mathbb{Z}_{2L}$ is ordered but this ordering is not compatible with the addition modulo $2L$ when $2L \geq 3$. When $2L \geq 3$, transformation (4.5) does not leave $\tilde{\mathcal{P}}_N^{2L}$ invariant.

We are particularly interested in operators realizing transformations that leave the parafermion algebra $\tilde{\mathcal{P}}_M^{2L}$ unchanged.

Definition 9 (Unitary symmetries). An operator $\tilde{U}$ is a symmetry if it is unitary, i.e.,

$$\tilde{U}^\dagger = \tilde{U}^{-1}, \quad (4.6a)$$

and the linear map

$$\begin{aligned} \text{Ad}_{\tilde{U}} : \tilde{\mathcal{P}}_N^{2L} &\longrightarrow \tilde{\mathcal{P}}_N^{2L}, \\ \tilde{A} &\longmapsto \tilde{U} \tilde{A} \tilde{U}^\dagger, \end{aligned} \quad (4.6b)$$

leaves the relations (3.1a)-(3.1c) invariant.

Example 8 (Parafermion charge-conjugation symmetry). The canonical parafermion charge conjugation $\tilde{U}_c$ that is defined by demanding that

$$\tilde{\gamma}_i \mapsto \tilde{U}_c \tilde{\gamma}_i \tilde{U}_c^\dagger := \tilde{\gamma}_i^\dagger, \quad i = 1, \dots, 2L, \quad (4.7)$$

generates a symmetry group of order two and is on-site factorizable according to **Lemma 5**.

Example 9 (Parafermion charge symmetry). Conjugation

$$\tilde{\gamma}_i \mapsto \tilde{U}_{\text{pf}} \tilde{\gamma}_i \tilde{U}_{\text{pf}}^\dagger := \omega_N \tilde{\gamma}_i, \quad i = 1, \dots, 2L, \quad (4.8)$$

by the global parafermion charge (4.2c) generates a symmetry group of order N and is on-site factorizable according to **Lemma 3**. We denote this symmetry group by $\mathbb{Z}_N^{\text{PF}}$.

Example 10 (Cyclic permutation composed with local charge conjugation). The linear map defined by

$$\tilde{\gamma}_1 \mapsto \tilde{\gamma}_1^\dagger, \quad \tilde{\gamma}_2 \mapsto \tilde{\gamma}_2, \quad \dots, \quad \tilde{\gamma}_{2L} \mapsto \tilde{\gamma}_{2L}, \quad (4.9)$$

acts inhomogeneously as it only acts non-trivially on the first site. This transformation does not leave the parafermion algebra $\tilde{\mathcal{P}}_N^{2L}$ invariant. However, the transformation

$$\tilde{\gamma}_1 \mapsto \tilde{\gamma}_{2L}^\dagger, \quad \tilde{\gamma}_2 \mapsto \tilde{\gamma}_1, \quad \dots, \quad \tilde{\gamma}_{2L} \mapsto \tilde{\gamma}_{2L-1}, \quad (4.10)$$

does leave the parafermion algebra $\tilde{\mathcal{P}}_N^{2L}$ invariant.

Let G_N^{2L} denote the set of all the transformations that can be represented linearly by conjugation with unitary operators from the parafermion algebra $\tilde{\mathcal{P}}_N^{2L}$ and leave the parafermion algebra $\tilde{\mathcal{P}}_N^{2L}$ invariant. We are going to study a subset G^{PF} of G_N^{2L} whose minimal content we now define.

Assumption 1 (Symmetries are on-site factorizable). In this thesis, the protecting symmetries are always taken to be unitary and on-site factorizable in a homogeneous way.

We postulate that the global parafermion charge symmetry from **Example 9** is always present. In other words, global parafermion charge is a conserved quantity. Moreover, we ask that the subgroup of symmetries associated to the global parafermion charge is closed under all symmetry transformations.

Assumption 2 (Parafermion global charge symmetry is a normal subgroup). Let G^{PF} be the total protecting symmetry group and $\mathbb{Z}_N^{\text{PF}}$ be the symmetry group associated to conservation of global parafermion charge. We assume that $\mathbb{Z}_N^{\text{PF}}$ is a normal subgroup of G^{PF} ,

$$\mathbb{Z}_N^{\text{PF}} \trianglelefteq G^{\text{PF}}. \quad (4.11)$$

Remark 10. **Assumption 2** is a weaker assumption than demanding that the global parafermion charge is independent from other symmetries, in which case $\tilde{U}_{\text{pf}}$ commutes with all other symmetries so that $\mathbb{Z}_N^{\text{PF}}$ is a central subgroup of G^{PF} ,

$$\mathbb{Z}_N^{\text{PF}} \leq G^{\text{PF}}. \quad (4.12)$$

By **Assumption 2**, the quotient $G^{\text{PF}}/\mathbb{Z}_N^{\text{PF}}$ forms a group. This allows us to view the protecting symmetries G^{PF} as a (general) group extension of the quotient $G^{\text{PF}}/\mathbb{Z}_N^{\text{PF}}$ by the parafermion charge symmetry $\mathbb{Z}_N^{\text{PF}}$, see **Appendix C**. This construction will allow us to separate the parafermionic structure $\mathbb{Z}_N^{\text{PF}}$ from the remaining (generic) symmetries captured by $G^{\text{PF}}/\mathbb{Z}_N^{\text{PF}}$. We cite the relevant results.

Fact 3 (Group extensions as Cartesian products). Let G^{PF} be a group with $\mathbb{Z}_N^{\text{PF}}$ a normal subgroup. Consider a pair of groups $\mathbb{Z}_N \cong \mathbb{Z}_N^{\text{PF}}$ and $G \cong G^{\text{PF}}/\mathbb{Z}_N^{\text{PF}}$. Then G^{PF} is a group extension of G by $\mathbb{Z}_N$. Moreover, there exists a group isomorphism

$$G^{\text{PF}} \cong (\mathbb{Z}_N \times G, \cdot), \quad (4.13a)$$

where the group operation ‘ $\cdot$ ’ is defined as

$$(z_1, g_1) \cdot (z_2, g_2) := (z_1 + \rho_{g_1}(z_2) + c(g_1, g_2), g_1 g_2), \quad \forall (z_1, g_1), (z_2, g_2) \in \mathbb{Z}_N \times G, \quad (4.13b)$$

for some group action $\rho_{(\cdot)}: G \rightarrow \text{Aut}(\mathbb{Z}_N)$ and 2-cocycle $c \in Z^2[G, \mathbb{Z}_N]$.

Remark 11. The group composition law ‘ $\cdot$ ’ is not commutative for a generic group action $\rho_{(\cdot)}: G \rightarrow \text{Aut}(\mathbb{Z}_N)$ since

$$(z_1, g_1) \cdot (z_2, g_2) \neq (z_2, g_2) \cdot (z_1, g_1) \quad \forall (z_1, g_1), (z_2, g_2) \in \mathbb{Z}_N \times G. \quad (4.14)$$

Remark 12 (Central and split extensions). When $\mathbb{Z}_N^{\text{PF}}$ is central in G^{PF} , we are considering *central* extensions. This is equivalent to the group action ρ being trivial. When G^{PF} can be written as a semi-direct product $\mathbb{Z}_N \rtimes G$, we talk of *split* extensions. This is equivalent to the 2-cocycle c vanishing.

Remark 13. • Consistency with the associativity of groups implies that c is a 2-cocycle, i.e., for all $g_1, g_2, g_3 \in G$

$$\rho_{g_1}(c(g_2, g_3)) - c(g_1 g_2, g_3) + c(g_1, g_2 g_3) - c(g_1, g_2) = 0 \quad (4.15a)$$

Moreover,

$$c(e, g) = c(g, e) = c(e, e) = 0 \quad \text{for all } g \in G. \quad (4.15b)$$

- Since ρ is a group action,

$$\rho_e = \text{Id} \in \text{Aut}(\mathbb{Z}_N), \quad (4.15c)$$

$$\rho_g(0) = 0 \quad \forall g \in G. \quad (4.15d)$$

Remark 14. The group structure of $(\mathbb{Z}_N \times G, \cdot)$ yields the following interpretations.

1. One verifies that

$$(z_1, e) \cdot (z_2, e) = (z_1 + z_2, e), \quad (4.16a)$$

$$(z_0, e) = (1, e)^{z_0}, \quad (4.16b)$$

$$(z, g) \cdot (z_0, e) \cdot (z, g)^{-1} = (\rho_g(z_0), e). \quad (4.16c)$$

Equations (4.16) show that the group $\mathbb{Z}_N \times \{e\}$ is isomorphic to $\mathbb{Z}_N$ and normal as a subgroup of $\mathbb{Z}_N \times G$. This justifies the identification of the element $(1, e) \in \mathbb{Z}_N \times G$ with the global parafermion charge operator that generates $\mathbb{Z}_N^{\text{PF}}$.

2. Equation (4.16c) lets us interpret the group action ρ as determining the change of the global parafermion charge operator under symmetry transformations. In fact, as is shown in **Lemma 13**, a group action of G on $\mathbb{Z}_N$ reduces to multiplication by coprimes modulo N . For all $g \in G$

$$\rho_g(z) \equiv a_g \cdot z \pmod{N} \quad \text{for some } a_g = 1, \dots, N-1, \quad \gcd(a_g, N) = 1. \quad (4.17)$$

3. For all $g_1, g_2 \in G$

$$(0, g_1) \cdot (0, g_2) = (c(g_1, g_2), g_1 g_2). \quad (4.18)$$

Accordingly, the 2-cocycle c quantifies the extent by which G^{PF} is not a direct product of $\mathbb{Z}_N^{\text{PF}}$ and $G^{\text{PF}}/\mathbb{Z}_N^{\text{PF}}$.

4. $(z, g) \in \mathbb{Z}_N \times G$,

$$(z, g) = (z, e) \cdot (e, g). \quad (4.19)$$

Identity (4.19) allows to factorize global parafermion parity from the symmetries in G . We emphasize that the order in the product matters on the right-hand side of Eq. (4.19) as the composition law ‘ $\cdot$ ’ need not be commutative.

Remark 15. We distinguish the normal subgroup $\mathbb{Z}_N^{\text{PF}} \trianglelefteq G^{\text{PF}}$ from the factor $\mathbb{Z}_N$ appearing in the Cartesian product $\mathbb{Z}_N \times G$. While they are isomorphic to one another, they are related via a non-unique isomorphism $G^{\text{PF}} \cong \mathbb{Z}_N \times G$.

4.3 Hamiltonians, boundary conditions, and boundary algebras

Definition 10 (Definition of ITP or SPT for parafermions). An invertible topological phase (ITP) without or with symmetry-protected topology (SPT) for parafermions is realized by continuous families of Hamiltonians satisfying **Assumptions 3-5** after the thermodynamic limit $2L \uparrow \infty$ at zero temperature has been taken.

Assumption 3 (Necessary conditions for symmetry-protected topology). Given is the pair of Hamiltonians

$$\widetilde{H}_{\mathbf{b}} = \widetilde{H}_{\mathbf{b}}^\dagger \in \widetilde{\mathcal{P}}_N^{2L}, \quad \mathbf{b} = 0, 1, \quad (4.20a)$$

that are distinguished by the boundary conditions they obey ($\mathbf{b} = 0$ selects periodic boundary conditions, $\mathbf{b} = 1$ selects open boundary conditions). It is assumed that

1. $\widehat{H}_{\mathbf{b}}$ is local in that it obeys the additive decomposition

$$\widetilde{H}_{\mathbf{b}} = \sum_{j=1+\mathbf{b}\ell}^{L-\mathbf{b}\ell} \widetilde{V}_j, \quad \widetilde{V}_j = \widetilde{V}_j^\dagger, \quad (4.20b)$$

where

$$\widetilde{V}_j \in \text{span}_{\mathbb{C}} \left\{ \prod_{n=j-\ell}^{j+\ell} (\widetilde{\gamma}_{2n-1}^{r_{2n-1}} \widetilde{\gamma}_{2n}^{r_{2n}}) \mid r_{2n-1}, r_{2n} = 0, 1, \dots, N-1, \right\} \quad (4.20c)$$

acts non trivially on an interval centered around j of some prescribed finite range 2ℓ ,

2. $\widehat{H}_{\mathbf{b}}$ has the symmetry group G^{PF} , whereby
 - (a) all elements of G^{PF} are represented linearly by unitary operators from the parafermion algebra $\widetilde{\mathcal{P}}_N^{2L}$ that are on-site factorizable in a homogeneous way,
 - (b) the global parafermion charge operator $\widetilde{U}_{\text{pf}}$ commutes with $\widehat{H}_{\mathbf{b}}$ and the group $\mathbb{Z}_N^{\text{PF}}$ it generates is a normal subgroup of G^{PF} ,
3. the ground state of $\widetilde{H}_{\mathbf{b}=0}$ is nondegenerate and gapped.

Question 1. Under **Assumption 3**, what is the minimal ground-state degeneracy of $\widetilde{H}_{\mathbf{b}=1}$ that is robust to local and symmetry-preserving perturbations that do not close the gap of $\widetilde{H}_{\mathbf{b}=0}$?

Answer 1. Any such protected ground-state degeneracy of $\widetilde{H}_{\mathbf{b}=1}$ must originate from boundary zero modes such that (i) their wave functions decay exponentially fast away from the left or right boundaries and (ii) they transform non-trivially under a projective representation of G^{PF} .

Assumption 4 (Existence of boundary zero modes). An open chain has two boundaries. We postulate that each one of the two boundaries hosts the integer number m of boundary zero modes that are responsible for the protected ground-state degeneracy of $\widetilde{H}_{\mathbf{b}=1}$.

Assumption 5 (Boundary parafermion algebra). Furthermore, we postulate that the m boundary zero modes on one of the two boundaries can be associated to the integer number m of parafermions $(\gamma_1, \dots, \gamma_m)$ and $(\gamma_1^\dagger, \dots, \gamma_m^\dagger)$ that generate the boundary parafermion algebra $\mathcal{P}_N^m$, whereby we assume that ²

$$\widetilde{U}_{\text{pf}} \gamma_i \widetilde{U}_{\text{pf}}^\dagger = \omega_N \gamma_i, \quad i = 1, \dots, m. \quad (4.22)$$

Remark 16. The relation between the bulk parafermions $(\tilde{\gamma}_1, \dots, \tilde{\gamma}_{2L})$ and the boundary parafermions $(\gamma_1, \dots, \gamma_m)$ depends on $\widetilde{H}_{\mathbf{b}}$. In this thesis, we just assume that such a relationship exists.

4.4 Bulk and boundary representations of the protecting symmetries G^{PF}

We consider realizations of the protecting symmetries that distinguish the bulk from the boundary. In particular, we allow the boundary realizations to be more relaxed than the

² We are thus ruling out the scenario by which the boundary zero modes have the larger parafermion charge $q_i = 2, \dots, N$

$$\widetilde{U}_{\text{pf}} \gamma_i \widetilde{U}_{\text{pf}}^\dagger = \omega_N^{q_i} \gamma_i, \quad i = 1, \dots, m, \quad (4.21)$$

where (assuming that N is even) $q_i = N/2$ realizes a fermion and $q_i = N$ realizes a boson. This situation can be thought of as the parafermionic version of anyon condensation [50]. Note that the N -th root of identity q_i is of order N if and only if $\gcd(q_i, N) = 1$. In this case, the generators of the boundary may still be interpreted as order- N parafermions.

global ones. This distinction is naturally captured by the use of linear representations in the bulk and projective representations at the boundary.

A sharp distinction between the bulk and the boundary is meaningful because of **Assumption 3**. By selecting PBCs, we isolate the bulk. The existence of a gapped ground state implies the existence of a natural length scale

$$\ell(\Delta) := \frac{\hbar v}{\Delta}, \quad (4.23)$$

where Δ is the gap between the ground state and all excited states v is the group velocity of the continuum of excitations just above the gap Δ , both defined after the thermodynamic limit $2L \uparrow \infty$ has been taken.³

The existence of the length scale $\ell(\Delta)$ is not tied to the degeneracy of the ground states. However, if the ground state is nondegenerate, it must necessarily transform trivially under the protecting symmetry group G^{PF} , i.e., it cannot transform under a non-trivial projective representation of the symmetry group G^{PF} according to **Lemma 8**. A necessary condition for the existence of a nondegenerate gapped ground state is thus that the symmetry G^{PF} be represented linearly.

Lemma 8 (Nondegeneracy excludes projective symmetry action on the ground state). Let H be a lattice Hamiltonian with symmetry group G , implemented by unitary operators $U(g)$, $g \in G$, satisfying

$$U(g) H U(g)^{-1} = H. \quad (4.24)$$

Assume that, after taking the thermodynamic limit, H has a nondegenerate, gapped ground state $|\Omega\rangle$, and that the symmetry G is not spontaneously broken. Then the action of G on the ground-state space is given by a linear (one-dimensional) representation of G . In particular, the ground state cannot transform according to a nontrivial projective representation of G .

Proof. Since H is invariant under G , the operators $U(g)$ map ground states to ground states. By nondegeneracy of the ground state, for each $g \in G$ there exists a phase $e^{i\theta(g)} \in \text{U}(1)$ such that

$$U(g) |\Omega\rangle = e^{i\theta(g)} |\Omega\rangle. \quad (4.25a)$$

Thus, the ground-state space carries a one-dimensional representation of G . Consider two group elements $g, h \in G$. On the one hand,

$$U(g) U(h) |\Omega\rangle = e^{i(\theta(g)+\theta(h))} |\Omega\rangle. \quad (4.25b)$$

On the other hand, by the group composition law,

$$U(gh) |\Omega\rangle = e^{i\theta(gh)} |\Omega\rangle. \quad (4.25c)$$

Since both expressions represent the action of the same symmetry element gh on the ground-state space, consistency requires

$$\theta(g) + \theta(h) - \theta(gh) = 0 \text{ mod } 2\pi. \quad (4.25d)$$

³The group velocity v can be computed numerically with the following scaling analysis. One fits the dependence of the energy difference between two consecutive excited states above a ground state on the number $2L$ of sites in the ring.

Hence $\theta : G \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ defines a group homomorphism, and the symmetry acts linearly on the ground-state subspace. A nontrivial projective representation would require a nontrivial Schur-multiplier $\omega(g, h) \in \text{U}(1)$ such that

$$U(g)U(h) = \omega(g, h)U(gh) \quad (4.25e)$$

with $\omega(g, h) \neq 1$. Such a factor system cannot be realized on a one-dimensional Hilbert space, since any phase ambiguity can be absorbed into a redefinition of the operators $U(g)$. Therefore, the ground state transforms according to a linear representation of G , and no nontrivial projective representation of G is possible on the ground-state subspace. $\square$

Assumption 3 presumes the existence of a stable ground-state degeneracy when OBCs are selected that originates from the left and right boundaries of an open chain. The boundary states have wave functions that decay exponentially fast with the distance $r\mathbf{a}$ ($\mathbf{a}$ is the lattice spacing) measured relative to the first or last site of the open chain. This exponential decay is proportional to the function

$$\exp\left(-\frac{r\mathbf{a}}{\ell(\Delta)}\right). \quad (4.26)$$

This means that (i) the boundary is not to be construed as a single point, but as a region whose characteristic size is of order $\ell(\Delta)$, and (ii) the ground-state degeneracy of a typical $\widetilde{H}_{\mathbf{b}=1}$ is only manifest in the thermodynamic limit, for the degeneracy below the energy gap Δ is lifted by energies proportional to

$$\exp\left(-2\frac{2L\mathbf{a}}{\ell(\Delta)}\right) \quad (4.27)$$

when the number $2L$ of sites of the open chain is finite. The stability of this degeneracy is conditioned by the locality of the Hamiltonians and by the boundary states realizing a non-trivial projective representation of the symmetry G^{PF} by application of **Lemma 8**.

The cartoon picture that captures the transition from bulk linear representations to boundary projective representation relies on the assumption that the symmetries are all on-site factorizable. This means that for any $(z, g) \in \mathbb{Z}_N \times G$, the global representation $\widetilde{U}[z, g]$ can be factorized into

$$\left(\widetilde{U}_1[z, g] \cdots \widetilde{U}_{\frac{\ell(\Delta)}{\mathbf{a}}}[z, g]\right) \omega[z, g] \omega[z, g]^\dagger \left(\text{bulk}\right) \omega[z, g] \omega[z, g]^\dagger \left(\widetilde{U}_{L-\frac{\ell(\Delta)}{\mathbf{a}}}[z, g] \cdots \widetilde{U}_L[z, g]\right), \quad (4.28a)$$

where the subscript j in $\widetilde{U}_j[z, g]$ means that its action is only non-trivial on site $j \in \{1, \dots, L\}$ of the chain. This suggests the identifications

$$U_L[z, g] = \omega[z, g] \left(U_1[z, g] \cdots U_m[z, g]\right) \quad (4.28b)$$

on the left boundary and

$$U_R[z, g] = \omega[z, g]^\dagger \left(U_{L-m+1}[z, g] \cdots U_L[z, g]\right) \quad (4.28c)$$

on the right boundary, where it is the ambiguity in choosing the Schur multiplier $\omega[z, g]$ (a unitary that belongs to the center of $\mathcal{P}_N^m$) that delivers the projective representations on the boundary.

4.4.1 Linear representations of G^{PF}

Definition 11 (Linear and unitary representation G^{PF}). A linear and unitary representation of the protecting symmetry group

$$G^{\text{PF}} \cong (\mathbb{Z}_N \times G, \bullet) \quad (4.29)$$

of the chain hosting the parafermion algebra $\mathcal{P}_N^{2L}$ is a group homomorphism

$$\begin{aligned} \tilde{U} : G^{\text{PF}} &\longrightarrow \tilde{\mathcal{P}}_N^{2L}, \\ (z, g) &\longmapsto \tilde{U}[(z, g)], \end{aligned} \quad (4.30a)$$

that fulfills the identities

$$\tilde{U}[(z, g)] \left\{ \tilde{U}[(z, g)] \right\}^\dagger = \mathbb{1}, \quad (4.30b)$$

$$\tilde{U}[(0, e)] = \mathbb{1}, \quad (4.30c)$$

$$\tilde{U}[(z_1, g_1)] \tilde{U}[(z_2, g_2)] = \tilde{U}[(z_1, g_1) \bullet (z_2, g_2)], \quad (4.30d)$$

for all $(z, g), (z_1, g_1), (z_2, g_2) \in G^{\text{PF}}$.

Remark 17 (Short-hand notation for $\tilde{U}$). When treating functions whose domain is the Cartesian product $\mathbb{Z}_N \times G$, we use the square bracket for their arguments $(z, g) \in \mathbb{Z}_N \times G$. We will often drop the parenthesis of elements (z, g) from $\mathbb{Z}_N \times G$ when they appear as stand-alone arguments, e.g.,

$$\tilde{U}[(z, g)] \equiv \tilde{U}[z, g]. \quad (4.31)$$

We will also use the compact short-hand notation

$$\left\{ \tilde{U}[(z, g)] \right\}^{-1} \equiv \tilde{U}[z, g]^{-1}, \quad \left\{ \tilde{U}[(z, g)] \right\}^\dagger \equiv \tilde{U}[z, g]^\dagger. \quad (4.32)$$

Assumption 6 (Global protecting symmetry is on-site factorizable). We assume that the protecting symmetry group G^{PF} is realized on the global parafermion algebra $\tilde{\mathcal{P}}_N^{2L}$ (i) linearly, (ii) unitarily, and (iii) in a homogeneous on-site factorizable way as in **Definition 8**.

4.4.2 Projective representations of G^{PF}

Both the dimensionality of the center $\mathcal{Z}(\mathcal{P}_N^m)$ of the parafermion algebra $\mathcal{P}_N^m$ and the existence of a global parafermion charge is tied to the parity of m . In particular, it is imperative to choose a Hilbert space whose dimensionality is the smallest one so as to accommodate a representation of the global parafermion charge. **Corollary 1** motivates the following definition.

Definition 12. Define the extended global parafermion algebra

$$\text{Ex}(\mathcal{P}_N^m) := \begin{cases} \mathcal{P}_N^m, & \text{if } m \text{ is even,} \\ \mathcal{P}_{N,\infty}^{m+1}, & \text{if } m \text{ is odd,} \end{cases} \quad (4.33)$$

where $\mathcal{P}_{N,\infty}^{m+1}$ is defined in Eq. (2.38b).

Assumption 7 (Boundary protecting symmetry is projective). We assume that the protecting symmetry group G^{PF} is realized unitarily and projectively on the extended global parafermion algebra $\text{Ex}(\mathcal{P}_N^m)$.

To refine what we mean by a projective representation, we make the following observation. Given is a realization $V[z, g] \in \mathcal{P}_N^m$ of an element $(z, g) \in \mathbb{Z}_N \times G$.⁴ Also given is $Z[z, g] \in \mathcal{Z}(\mathcal{P}_N^m)$ assumed to be unitary. Define

$$U[z, g] := Z[z, g] V[z, g]. \quad (4.34)$$

We then have the identity

$$\begin{aligned} U[z, g] \gamma_i U[z, g]^\dagger &= \left(Z[z, g] V[z, g] \right) \gamma_i \left(Z[z, g]^\dagger V[z, g]^\dagger \right) \\ &\stackrel{\text{commutativity}}{=} \left(V[z, g] Z[z, g] \right) \gamma_i \left(Z[z, g]^\dagger V[z, g]^\dagger \right) \\ &\stackrel{\text{associativity}}{=} V[z, g] \left(Z[z, g] \gamma_i Z[z, g]^\dagger \right) V[z, g]^\dagger \\ &\stackrel{\text{commutativity}}{=} V[z, g] \left(\gamma_i Z[z, g] Z[z, g]^\dagger \right) V[z, g]^\dagger \\ &\stackrel{\text{unitarity}}{=} V[z, g] \gamma_i V[z, g]^\dagger, \end{aligned} \quad (4.35)$$

where $i = 1, \dots, m$. We have shown that multiplication by central factors of any operator that realizes a unitary element of G^{PF} leaves the action (by conjugation) of this operator unchanged. Equation (4.35) motivates the following two definitions.

Definition 13 (Projective representation of G^{PF}). A projective representation of the protecting symmetry group

$$G^{\text{PF}} \cong (\mathbb{Z}_N \times G, \bullet) \quad (4.36a)$$

on a boundary hosting the parafermion algebra $\mathcal{P}_N^m$ is a map

$$\begin{aligned} U : G^{\text{PF}} &\longrightarrow \text{Ex}(\mathcal{P}_N^m), \\ (z, g) &\longmapsto U[(z, g)], \end{aligned} \quad (4.36b)$$

such that

$$U[(z, g)] U[(z, g)]^\dagger = \mathbb{1}, \quad (4.36c)$$

$$U[(z_1, g_1)] U[(z_2, g_2)] = \Phi[(z_1, g_1), (z_2, g_2)] U[(z_1, g_1) \bullet (z_2, g_2)], \quad (4.36d)$$

for all $(z, g), (z_1, g_1), (z_2, g_2) \in G^{\text{PF}}$. Here, the Schur-multiplier

$$\Phi : G^{\text{PF}} \times G^{\text{PF}} \longrightarrow \mathcal{Z}(\mathcal{P}_N^m) \quad (4.36e)$$

is a map Φ taking values in the center $\mathcal{Z}(\mathcal{P}_N^m)$ of the parafermion algebra $\mathcal{P}_N^m$ on the boundary that is compatible with the associativity of $\text{Ex}(\mathcal{P}_N^m)$.

⁴We use the short-hand notations $V[(z, g)] \equiv V[z, g]$ and $\{V[(z, g)]\}^\dagger \equiv V[z, g]^\dagger$.

Remark 18 (Short-hand notation for U). When treating functions whose domain is the Cartesian product $\mathbb{Z}_N \times G$, we use the square bracket for their arguments $(z, g) \in \mathbb{Z}_N \times G$. We will often drop the parenthesis of elements (z, g) from $\mathbb{Z}_N \times G$ when they appear as stand-alone arguments, e.g.,

$$U[(z, g)] \equiv U[z, g]. \quad (4.37)$$

We will also use the compact short-hand notation

$$\{U[(z, g)]\}^{-1} \equiv U[z, g]^{-1}, \quad \{U[(z, g)]\}^\dagger \equiv U[z, g]^\dagger. \quad (4.38)$$

Remark 19. Equation (4.36d) states that projective representations are group homomorphism up to central factors. The Schur-multiplier Φ characterizes the extent to which a projective representation fails to be a linear one. To be consistent with the associativity of the algebra $\mathcal{P}_N^m$, the Schur-multiplier Φ must be 2-cocycle as defined in **Appendix B** and **D**.

We define an equivalence relation on projective representations. Two projective representations are considered to be equivalent, if they differ only by central factors.

Definition 14 (Projective equivalence). Two projective boundary representations U and U' G^{PF} are projectively equivalent if there exists a map

$$\Theta : G^{\text{PF}} \longrightarrow \mathcal{Z}(\mathcal{P}_N^m) \quad (4.39a)$$

that satisfies

$$U'[z, g] = \Psi[z, g] U[z, g] \quad (4.39b)$$

for any $(z, g) \in \mathbb{Z}_N \times G$, i.e., it is a 1-cochain,

$$\Theta \in C^1[G^{\text{PF}}, \mathcal{Z}(\mathcal{P}_N^m)]. \quad (4.39c)$$

Remark 20. Θ can be thought of as a gauge map.

Fact 4. **Definition 14** yields an equivalence relation on the set of projective representations. One may show that when m is even, a pair projective representations are equivalent if and only if their Schur-multipliers differ by a 2-coboundary. This leads to a classification of the equivalence classes of projective representations given by the second group cohomology

$$H^2[G^{\text{PF}}, \mathcal{Z}(\mathcal{P}_N^m)] \quad (4.40)$$

when m is even.

Remark 21. When m is odd, the Schur-multipliers do not necessarily commute with the symmetry representations. A twisted version of group cohomology is needed in this case, which can also be used to study anti-unitary projective representations, see *Remark 38* and **Appendix D** for more details.

4.4.3 Compatibility between boundary and bulk representations

As we can express any of the m generators of the parafermion algebra $\mathcal{P}_N^m$ in terms of the $2L$ generators of the bulk parafermion algebra $\tilde{\mathcal{P}}_N^{2L}$, we must impose the compatibility conditions

$$U[z, g] \gamma_i U[z, g]^\dagger \stackrel{!}{=} \tilde{U}[z, g] \gamma_i \tilde{U}[z, g]^\dagger, \quad i = 1, \dots, m, \quad (4.41)$$

for any $(z, g) \in \mathbb{Z}_N \times G$. Equation dictates that (i) the global and boundary symmetries are the same (the possibility of spontaneous symmetry breaking is excluded) and (ii) the transformation laws of the generators of the boundary parafermion algebra under the protecting global symmetries are the same as under the protecting boundary symmetries.

Particular care is needed in the odd- m case for which an additional auxiliary parafermion γ_∞ is added. The parafermion γ_∞ appears only in the extended boundary algebra $\mathcal{P}_{N,\infty}^{m+1}$. It originates from the opposite boundary of the chain, infinitely far away after the thermodynamic limit $2L \uparrow \infty$ has been taken. From the point of view of the boundary $\mathcal{P}_{N,\infty}^m$, information on the bulk and the opposite boundary has been lost. For this reason, we relax condition (4.41) when applied to γ_∞ in that its transformation laws under the protecting global symmetries may differ from its transformation laws under the protecting boundary symmetries by a multiplicative phase factors.

Assumption 8. Given are the bulk representation $\tilde{U}$ under **Assumption 6** and the boundary representation U under **Assumption 7** of the protecting symmetry $G^{\text{PF}} \cong (\mathbb{Z}_N \times G, \cdot)$. For any $(z, g) \in \mathbb{Z}_N \times G$, we assume that

$$U[z, g] \gamma_i U[z, g]^\dagger \stackrel{!}{=} \tilde{U}[z, g] \gamma_i \tilde{U}[z, g]^\dagger, \quad i = 1, \dots, m, \quad (4.42a)$$

for any integer m , while we assume that

$$U[z, g] \gamma_\infty U[z, g]^\dagger \stackrel{!}{=} e^{i\tilde{\varepsilon}[z, g]} \tilde{U}[z, g] \gamma_\infty \tilde{U}[z, g]^\dagger, \quad (4.42b)$$

for some function

$$\begin{aligned} \tilde{\varepsilon} : \mathbb{Z}_N \times G &\longrightarrow \mathbb{R}/2\pi\mathbb{Z}, \\ (z, g) &\longmapsto \tilde{\varepsilon}[z, g], \end{aligned} \quad (4.42c)$$

when m is an odd integer.

First, recall that the standard parafermion charge operator on $\mathcal{P}_{N,\infty}^{m+1}$ is

$$U_{\text{pf}} \equiv \left(\prod_{j=1}^{\frac{m-1}{2}} \omega_N^{S_N} \gamma_{2j-1} \gamma_{2j}^\dagger \right) (\omega_N^{S_N} \gamma_m \gamma_\infty^\dagger) \equiv Y \gamma_\infty^\dagger \quad (4.43)$$

by **Definition 4**. Second, according to Eq. (4.16c), conjugation by $U[z, g]$ of U_{pf} must change the unit of parafermion charge from 1 to $\rho_g(1)$. Third, symmetries are on-site factorizable by **Assumption 1**. These observations justify the following final assumption.

Assumption 9. Given is the boundary representation U under **Assumption 7** of the protecting symmetry $G^{\text{PF}} \cong (\mathbb{Z}_N \times G, \bullet)$ when m is odd. We shall assume that

$$U[z, g] \gamma_\infty U[z, g]^\dagger \stackrel{!}{=} \omega_N^{k_\infty[z, g]} \gamma_\infty^{\rho_g(1)}, \quad k_\infty[z, g] \in \mathbb{Z}_N, \quad (4.44a)$$

and

$$U[z, g] Y U[z, g]^\dagger \stackrel{!}{=} e^{i\tilde{\sigma}[z, g]} Y^{\rho_g(1)}, \quad \tilde{\sigma}[z, g] \in \mathbb{R}/2\pi\mathbb{Z}, \quad (4.44b)$$

for any $(z, g) \in \mathbb{Z}_N \times G$.

If we combine Eq. (4.43) with **Assumption 9**, we infer that

$$U[z, g] U_{\text{pf}} U[z, g]^\dagger = e^{i\tilde{\sigma}[z, g]} \omega_N^{-k_\infty[z, g]} U_{\text{pf}}^{\rho_g(1)}. \quad (4.45)$$

As desired, conjugation of the standard parafermion charge operator U_{pf} by $U[z, g]$ changes the unit of parafermion charge from 1 to $\rho_g(1)$ as the image of U_{pf} is $U_{\text{pf}}^{\rho_g(1)}$, up to a multiplicative phases factor.

Chapter 5

Classification of parafermionic SPTs

5.1 Introduction

In **Chapter 5**, we carry out the classification of symmetry-protected topological phases for one-dimensional parafermion systems. Our goal is to translate the assumptions introduced in **Chapter 4** into concrete algebraic invariants, and to formulate a classification scheme that cleanly isolates the parafermionic charge contribution.

We begin in **Section 5.2** by fixing the parafermion chain, its protecting symmetry group, and the class of admissible Hamiltonians and boundary algebras.

The classification of parafermionic SPTs is then reduced in **Section 5.3** to the analysis of projective representations of the protecting symmetry group on the boundary, subject to the compatibility conditions with the bulk symmetry action.

A key structural distinction arises between even and odd boundary parafermion number. In the even case discussed in **Subsection 5.3.1**, boundary representations are classified by ordinary $U(1)$ -valued Schur-multipliers together with an additional index measuring the obstruction to implementing charge conjugation as a linear symmetry. In the odd case discussed in **Subsection 5.3.2**, the presence of a non-trivial center leads to a broadened notion of projectivity, involving both $U(1)$ - and $\mathbb{Z}_N$ -valued data.

Finally, we show how these invariants decompose systematically, yielding a transparent parametrization of equivalence classes of boundary representations. This provides a conceptually clean classification of parafermionic SPT phases within the stated assumptions.

5.2 Setup

5.2.1 One-dimensional chain of parafermions

Let $L \gg 1, N \geq 2$ be two non-vanishing integers. As in **Chapter 4**, we consider an ordered one-dimensional lattice

$$\Lambda = \{1, 2, \dots, 2L\} \tag{5.1}$$

consisting of an even number $2L$ of sites. Each site $i \in \Lambda$ is assigned the order- N parafermion operator $\tilde{\gamma}_i$ and its adjoint $\tilde{\gamma}_i^\dagger$ and the pair of sets $(\tilde{\gamma}_1, \dots, \tilde{\gamma}_{2L})$ and $(\tilde{\gamma}_1^\dagger, \dots, \tilde{\gamma}_{2L}^\dagger)$ obey the

relations defined in Eqs. (2.2a)-(2.2c). We may thus assign to Λ the parafermion algebra

$$\tilde{\mathcal{P}}_N^{2L} := \text{span}_{\mathbb{C}} \left\{ \tilde{\gamma}_1^{k_1} \tilde{\gamma}_2^{k_2} \cdots \tilde{\gamma}_{2L}^{k_{2L}} \mid k_1, \dots, k_{2L} \in \mathbb{Z}_N \right\} \quad (5.2)$$

generated by $2L$ order- N parafermions as studied in **Chapter 3**. We have reserved the symbol $\tilde{\cdot}$ to emphasize that the parafermion operators act on the entire one-dimensional chain. When considering a boundary parafermion operator whose support is only on an appropriately defined boundary, we shall drop the symbol $\tilde{\cdot}$.

As there are an even number $2L$ of sites in Λ , we may group pairs of neighboring sites into L unit cells labeled by $j = 1, \dots, L$. As shown in **Chapter 3**, each unit cell $j = 1, \dots, L$ carries the local parafermion parity operator

$$\tilde{U}_{\text{pf},j} = \omega_N^{S_N} \tilde{\gamma}_{2j-1} \tilde{\gamma}_{2j}^\dagger. \quad (5.3)$$

Moreover, there exists the global parafermion parity operator

$$\tilde{U}_{\text{pf}} = \prod_{j=1}^L \tilde{U}_{\text{pf},j} = \prod_{j=1}^L \omega_N^{S_N} \tilde{\gamma}_{2j-1} \tilde{\gamma}_{2j}^\dagger. \quad (5.4)$$

By **Proposition 3**, the lattice Λ supports a $\mathbb{Z}_N^{\text{PF}}$ -graded Hilbert space $\mathcal{H}$. The global parafermion parity operator is on-site factorizable, see **Definition 8**.

5.2.2 Protecting symmetries

The protecting symmetries were described in **Chapter 4.4**. They form the group G^{PF} containing the order- N parafermion parity symmetry $\mathbb{Z}_N^{\text{PF}}$ as a normal subgroup,

$$\mathbb{Z}_N^{\text{PF}} \trianglelefteq G^{\text{PF}}. \quad (5.5)$$

The total protecting symmetries may be constructed as a group extension of a group $G \cong G^{\text{PF}}/\mathbb{Z}_N^{\text{PF}}$ by $\mathbb{Z}_N^{\text{PF}}$. We have the group isomorphism

$$G^{\text{PF}} \cong (\mathbb{Z}_N \times G, \bullet), \quad (5.6a)$$

where the group operation ' $\bullet$ ' is defined as

$$(z_1, g_1) \bullet (z_2, g_2) := (z_1 + \rho_{g_1}(z_2) + c(g_1, g_2), g_1 g_2), \quad (5.6b)$$

for some group action $\rho_{(\cdot)} : G \rightarrow \text{Aut}(\mathbb{Z}_N)$ and 2-cocycle $c \in Z^2[G, \mathbb{Z}_N]$. Computations in this chapter will be performed on $\mathbb{Z}_N \times G$ using this isomorphism.

The factorization of G^{PF} (5.6a) yields the following partition and identity on the group.

Lemma 9 (Partition and identities for the group extension G^{PF}). Given is the group extension of G by $\mathbb{Z}_N^{\text{PF}}$, such that $G^{\text{PF}} \cong (\mathbb{Z}_N \times G, \bullet)$ is the Cartesian product $\mathbb{Z}_N \times G$ as a set with the group composition ' $\bullet$ ' defined in Eq. (5.6), say. The following three properties will be useful.

1. As a set, the Cartesian product $\mathbb{Z}_N \times G$ can be partitioned into the disjoint union

$$\begin{aligned} \mathbb{Z}_N^{\text{PF}} \times G = & \{(0, e), (1, e)\} \\ & \bigsqcup \{(z, e) \mid z \in \mathbb{Z}_N^{\text{PF}} \setminus \{0, 1\}\} \\ & \bigsqcup \{(0, g) \mid g \in G \setminus \{e\}\} \\ & \bigsqcup \{(z, g) \mid z \in \mathbb{Z}_N^{\text{PF}} \setminus \{0\}, g \in G \setminus \{e\}\}. \end{aligned} \quad (5.7a)$$

2. Normality of $\mathbb{Z}_N$ may be expressed as the identity

$$(z, g) \cdot (z_0, e) \cdot (z, g)^{-1} = (\rho_g(z_0), e) = (\rho_g(1), e)^{z_0} \quad (5.7b)$$

for any $z_0 \in \mathbb{Z}_N$ and for any $(z, g) \in \mathbb{Z}_N \times G$.

3. Generic group elements $(z, g) \in \mathbb{Z}_N \times G$ factorize as

$$(z, g) = (z, e) \cdot (0, g). \quad (5.7c)$$

Proof. The partition (5.7a) may be verified by counting the number of elements in each set. Identities (5.7b) and (5.7c) follow from the group multiplication rule (5.6b). $\square$

For convenience, we recall the content of **Assumption 6**. The global protecting symmetry $G^{\text{PF}} \cong (\mathbb{Z}_N \times G, \cdot)$ is represented by a map

$$\begin{aligned} \tilde{U} : \mathbb{Z}_N \times G &\longrightarrow \tilde{\mathcal{P}}_N^{2L}, \\ (z, g) &\longmapsto \tilde{U}[z, g], \end{aligned} \quad (5.8)$$

that is (i) a linear representation, (ii) unitary, and (iii) on-site factorizable in a homogeneous way. Condition (i) means that $\tilde{U}$ is a group homomorphism. As such it preserves the group structure of $G^{\text{PF}} \cong (\mathbb{Z}_N \times G, \cdot)$, i.e.,

$$\tilde{U}[z_1, g_1] \tilde{U}[z_2, g_2] = \tilde{U}[(z_1, g_1) \cdot (z_2, g_2)] \quad (5.9)$$

for all pairs $(z_1, g_1), (z_2, g_2) \in \mathbb{Z}_N \times G$. Condition (ii) means that

$$\tilde{U}[z, g] = \tilde{U}[z, g]^\dagger \quad (5.10)$$

for all pairs $(z, g) \in \mathbb{Z}_N \times G$. Condition (iii) means that for any $(z, g) \in \mathbb{Z}_N \times G$, there exists the local functional

$$\begin{aligned} U_{z,g} : \mathbb{Z}_N \times G &\longrightarrow \mathcal{P}_N^2 \equiv \text{span}_{\mathbb{C}} \left\{ \xi^{k_1} \eta^{k_2} \mid k_1, k_2 \in \mathbb{Z}_N \right\}, \\ (z, g) &\longmapsto U_{z,g}[\xi, \eta] = U_{z,g}[\xi, \eta]^\dagger, \end{aligned} \quad (5.11a)$$

such that ¹

$$\tilde{U}[z, g] = U_{z,g}[\tilde{\xi}_1, \tilde{\eta}_1] \cdots U_{z,g}[\tilde{\xi}_L, \tilde{\eta}_L], \quad (5.11b)$$

$$\left[U_{z_1, g_1}[\tilde{\xi}_j, \tilde{\eta}_j], U_{z_2, g_2}[\tilde{\xi}_l, \tilde{\eta}_l] \right] = 0, \quad j, l = 1, \dots, L, \quad (5.11c)$$

$$U_{z_1, g_1}[\tilde{\xi}_j, \tilde{\eta}_j] U_{z_2, g_2}[\tilde{\xi}_j, \tilde{\eta}_j] = U_{(z_1, g_1) \cdot (z_2, g_2)}[\tilde{\xi}_j, \tilde{\eta}_j], \quad j = 1, \dots, L, \quad (5.11d)$$

¹The notation $U_{z,g}[\xi, \eta]$ is used to emphasize the condition of homogeneity. The same functional is used for all sites $j = 1, \dots, L$. The argument of the functional distinguishes the sites.

for all $(z, g), (z_1, g_1), (z_2, g_2) \in \mathbb{Z}_N \times G$. Finally, we make the identification

$$\tilde{U}[1, e] = \tilde{U}_{\text{pf}}, \quad (5.11e)$$

where $\tilde{U}_{\text{pf}}$ is defined in Eq. (5.4).

5.2.3 Hamiltonians and their boundary degrees of freedom

As in **Chapter 4**, we are given is the pair of Hamiltonians

$$\tilde{H}_{\mathbf{b}} = \tilde{H}_{\mathbf{b}}^\dagger \in \tilde{\mathcal{P}}_N^{2L}, \quad \mathbf{b} = 0, 1, \quad (5.12)$$

that are distinguished by the boundary conditions they obey, namely $\mathbf{b} = 0$ selects PBCs while $\mathbf{b} = 1$ select OBCs. **Assumption 3** is obeyed by $\tilde{H}_{\mathbf{b}}$ for $\mathbf{b} = 0, 1$, whereby the boundary degrees of freedom associated to the ground-state degeneracy of $\tilde{H}_{\mathbf{b}=1}$ can be associated to an integer number $m \geq 1$ of parafermions $(\gamma_1, \dots, \gamma_m)$. These boundary parafermions are assumed to have global parity one,

$$\tilde{U}_{\text{pf}} \gamma_i \tilde{U}_{\text{pf}}^\dagger = \omega_N \gamma_i, \quad i = 1, \dots, m. \quad (5.13)$$

We are given the (left) boundary algebra

$$\mathcal{P}_N^m = \text{span}_{\mathbb{C}} \left\{ \gamma_1^{k_1} \dots \gamma_m^{k_m} \mid k_1, \dots, k_m = 0, 1, \dots, N-1 \right\}. \quad (5.14)$$

The relation between the bulk parafermions $(\tilde{\gamma}_1, \dots, \tilde{\gamma}_{2L})$ and the boundary parafermions $(\gamma_1, \dots, \gamma_m)$ depends on $\tilde{H}_{\mathbf{b}}$. In this chapter, we just assume that such a relationship exists. It is the absence of the symbol $\sim$ that will imply that we consider the boundary in isolation.

5.2.4 Boundary representations when m is even

Even- m boundary parafermion algebra

When m is even, the boundary parafermion algebra $\mathcal{P}_N^m$ fulfills all the properties demonstrated in **Chapter 3**. According to **Proposition 1**, the center is trivial and given by

$$\mathcal{Z}(\mathcal{P}_N^m) = \text{span}_{\mathbb{C}} \{1\}. \quad (5.15)$$

According to **Proposition 2**, any unitary operator realizing the boundary parafermion charge is of the form

$$e^{i\gamma} U_{\text{pf}}, \quad \gamma \in [0, 2\pi), \quad U_{\text{pf}} = \prod_{j=1}^{m/2} \omega_N^{S_N} \gamma_{2j-1} \gamma_{2j}^\dagger. \quad (5.16)$$

The multiplicative phase factor $e^{i\gamma}$ is an example of a $U(1)$ -redundancy when defining unitary operators on the boundary.

Projective representations on the even- m boundary

Given this $U(1)$ -redundancy, we define the projective representations on the even- m boundary as follows.

Definition 15 (Even- m projective representations and their equivalences). Given is the even- m boundary parafermion algebra $\mathcal{P}_N^m$ with the protecting symmetry group $G^{\text{PF}} \cong (\mathbb{Z}_N \times G)$. A projective boundary representation of the protecting symmetry group G^{PF} with $U(1)$ -valued Schur-multipliers is a map

$$\begin{aligned} U : (\mathbb{Z}_N \times G) &\longrightarrow \mathcal{P}_N^m, \\ (z, g) &\longmapsto U[z, g], \end{aligned} \quad (5.17a)$$

such that

$$U[z_1, g_1] U[z_2, g_2] = e^{i\phi[(z_1, g_1), (z_2, g_2)]} U[(z_1, g_1) \cdot (z_2, g_2)], \quad (5.17b)$$

where

$$\phi : (\mathbb{Z}_N^{\text{PF}} \times G)^2 \rightarrow \mathbb{R}/2\pi\mathbb{Z} \quad (5.17c)$$

is compatible with the associativity of the operator multiplication.

We say that two projective representations U and U' are projectively equivalent ($U \sim U'$) if there exists a map

$$\theta : (\mathbb{Z}_N \times G) \rightarrow \mathbb{R}/2\pi\mathbb{Z} \quad (5.18)$$

such that

$$U'[z, g] = e^{i\theta[z, g]} U[z, g] \quad (5.19)$$

for all $(z, g) \in (\mathbb{Z}_N \times G)$.

Remark 22 ($\mathbb{R}/2\pi\mathbb{Z} \cong U(1)$). The additive group $\mathbb{R}/2\pi\mathbb{Z}$ is isomorphic to the multiplicative group $U(1) := \{u \in \mathbb{C} \mid |u| = 1\}$.

Remark 23. One may show that

1. the associativity of the parafermion algebra $\mathcal{P}_N^m$ implies that ϕ is a 2-cocycle, $\phi \in Z^2[G^{\text{PF}}, U(1)]$,
2. two projective representations are equivalent if and only if they differ by a 2-coboundary, i.e.,

$$U \sim U' \iff U - U' = \delta^1 \theta \quad (5.20)$$

for some 1-cochain $\theta \in C^1[G^{\text{PF}}, U(1)]$.

The projective boundary representations when m is even as in **Definition 15** are thus classified by the second group cohomology $H^2[G^{\text{PF}}, U(1)]$.

Compatibility between boundary and bulk representations

Consistency demands that the action on the boundary of the global ($\tilde{U}$) and boundary (U) representations match, i.e., for any $(z, g) \in \mathbb{Z}_N \times G$ condition

$$U[z, g], \gamma_i U[z, g]^\dagger \stackrel{!}{=} \tilde{U}[z, g] \gamma_i \tilde{U}[z, g]^\dagger \quad (5.21)$$

must hold for $i = 1, \dots, m$. This is the content of **Assumption 8**.

5.2.5 Boundary representations when m is odd

Odd- m boundary parafermion algebra

When m is odd, the boundary algebra $\mathcal{P}_N^m$ has a non-trivial center. According to **Proposition 1**, the center

$$\mathcal{Z}(\mathcal{P}_N^m) = \text{span}_{\mathbb{C}}\{Y^k \mid k \in \mathbb{Z}_N\}, \quad (5.22)$$

is generated by the unitary order- N operator

$$Y \equiv \lambda_N \left(\prod_{j=1}^{\frac{m-1}{2}} \gamma_{2j-1} \gamma_{2j}^\dagger \right) \gamma_m, \quad \lambda_N \in \text{U}(1). \quad (5.23)$$

In particular, the boundary algebra does not support a parafermion charge operator, as was shown in **Proposition 2**.

Following **Corollary 1** and **Chapter 4.4.2**, we consider an extension of the m -odd boundary parafermion algebra $\mathcal{P}_N^m$ by adding an auxiliary parafermion generator γ_∞ to the m -odd generators $\{\gamma_1, \dots, \gamma_m\}$. In doing so, we have embedded $\mathcal{P}_N^m$ into $\mathcal{P}_{N,\infty}^{m+1}$. This embedding delivers the parafermion charge operator

$$U_{\text{pf}} = e^{i\alpha} \left(\prod_{j=1}^{\frac{m-1}{2}} \omega_N^{S_N} \gamma_{2j-1} \gamma_{2j}^\dagger \right) (\omega_N^{S_N} \gamma_m \gamma_\infty^\dagger), \quad \alpha \in \mathbb{R}/2\pi\mathbb{Z}, \quad (5.24)$$

that assigns to each of $\{\gamma_1, \dots, \gamma_m, \gamma_\infty\}$ the $(m+1)$ -even parafermion charge one. However, according to **Corollary 2**, had we used the unitary operator

$$e^{i\alpha} Y^k U_{\text{pf}} \equiv e^{i\alpha} Y^{k+1} \gamma_\infty^\dagger, \quad k \in \mathbb{Z}_N, \quad (5.25)$$

to measure the parafermion charge in $\mathcal{P}_{N,\infty}^{m+1}$ we would still have found the parafermion charge one for the m -odd $\{\gamma_1, \dots, \gamma_m\}$ generators, while we would have found the charge k for the auxiliary parafermion γ_∞ . This means that the construction of a parafermion charge operator that assigns a sharp parafermion charge to all $(m+1)$ -even $\{\gamma_1, \dots, \gamma_m, \gamma_\infty\}$ generators that is one for the odd- m generators $\{\gamma_1, \dots, \gamma_m\}$ and $\mathbb{Z}_N$ -valued for γ_∞ is not unique. There is an ambiguity isomorphic to $(\text{U}(1) \times \mathbb{Z}_N)$ in doing so.

Projective representations on the odd- m boundary

Given this $(\text{U}(1) \times \mathbb{Z}_N)$ -ambiguity, we define the projective representations on the odd- m boundary as follows.

Definition 16 (Odd- m projective representations and their equivalences). Given is the odd- m boundary extended parafermion algebra $\mathcal{P}_N^m \subset \mathcal{P}_{N,\infty}^{m+1}$ with the protecting symmetry group $G^{\text{PF}} \cong (\mathbb{Z}_N \times G)$. A projective boundary representation of the protecting symmetry group G^{PF} with $(\text{U}(1) \times \mathbb{Z}_N)$ -valued Schur-multipliers is a map

$$\begin{aligned} U : (\mathbb{Z}_N \times G) &\longrightarrow \mathcal{P}_{N,\infty}^{m+1}, \\ (z, g) &\longmapsto U[z, g], \end{aligned} \quad (5.26a)$$

such that

$$U[z_1, g_1] U[z_2, g_2] = e^{i\phi_1[(z_1, g_1), (z_2, g_2)]} Y^{\phi_2[(z_1, g_1), (z_2, g_2)]} U[(z_1, g_1) \cdot (z_2, g_2)], \quad (5.26b)$$

where

$$\phi_1 : (\mathbb{Z}_N \times G)^2 \rightarrow \mathbb{R}/2\pi\mathbb{Z}, \quad (5.26c)$$

$$\phi_2 : (\mathbb{Z}_N \times G)^2 \rightarrow \mathbb{Z}_N, \quad (5.26d)$$

are both compatible with the associativity of the operator multiplication.

We say that two projective representations U and U' are projectively equivalent ($U \sim U'$) if there exists a pair of maps

$$\theta_1 : (\mathbb{Z}_N \times G) \rightarrow \mathbb{R}/2\pi\mathbb{Z}, \quad (5.27a)$$

$$\theta_2 : (\mathbb{Z}_N \times G) \rightarrow \mathbb{Z}_N, \quad (5.27b)$$

such that

$$U'[z, g] = e^{i\theta_1[z, g]} Y^{\theta_2[z, g]} U[z, g] \quad (5.28)$$

for all $(z, g) \in (\mathbb{Z}_N \times G)$.

Remark 24. The additive group $\mathbb{R}/2\pi\mathbb{Z}$ is isomorphic to the multiplicative group $U(1) := \{u \in \mathbb{C} \mid |u| = 1\}$.

Remark 25. The associativity of the parafermion algebra $\mathcal{P}_{N,\infty}^{m+1}$ implies compatibility conditions on the $U(1) \times \mathbb{Z}_N$ -valued Schur-multiplier that generalize 2-cocycles to twisted 2-cocycles and 2-coboundaries to twisted 2-coboundaries. One must then use twisted cohomology to obtain an algebraic classification of the projective equivalent boundary representations, as the central element Y need not commute with any symmetry operator $U[z, g] \in \mathcal{P}_{N,\infty}^{m+1}$.

Compatibility between boundary and bulk representations

Consistency demands that the action of the global ($\tilde{U}$) and boundary (U) representations match on the parafermion algebra $\mathcal{P}_N^m$, i.e., for any $(z, g) \in \mathbb{Z}_N \times G$ condition

$$U[z, g] \gamma_i U[z, g]^\dagger \stackrel{!}{=} \tilde{U}[z, g] \gamma_i \tilde{U}[z, g]^\dagger \quad (5.29a)$$

must hold for $i = 1, \dots, m$, while condition

$$U[z, g] \gamma_\infty U[z, g]^\dagger \stackrel{!}{=} e^{i\tilde{\varepsilon}[z, g]} \tilde{U}[z, g] \gamma_\infty \tilde{U}[z, g]^\dagger \quad (5.29b)$$

for some function

$$\begin{aligned} \tilde{\varepsilon} : \mathbb{Z}_N \times G &\longrightarrow \mathbb{R}/2\pi\mathbb{Z}, \\ (z, g) &\longmapsto \tilde{\varepsilon}[z, g], \end{aligned} \quad (5.29c)$$

must hold for the auxiliary parafermion γ_∞ , as stated in **Assumption 8**. Hereto, we make the additional **Assumption 9**, namely there exists the pair of functions $k_\infty : \mathbb{Z}_N \times G \longrightarrow \mathbb{Z}_N$ and $\tilde{\sigma} : \mathbb{Z}_N \times G \longrightarrow \mathbb{Z}/2\pi\mathbb{Z}$ such that

$$U[z, g] \gamma_\infty U[z, g]^\dagger \stackrel{!}{=} \omega_N^{k_\infty[z, g]} \gamma_\infty^{\rho_g(1)}, \quad (5.30a)$$

$$U[z, g] Y U[z, g]^\dagger \stackrel{!}{=} e^{i\tilde{\sigma}[z, g]} Y^{\rho_g(1)}, \quad (5.30b)$$

for all $(z, g) \in \mathbb{Z}_N \times G$.

5.3 Classification of boundary representations of protecting symmetries

As established in **Chapters 2 and 3**, the boundary algebras $\mathcal{P}_N^m$ generated by m parafermions differ significantly depending on whether m is even or odd. Hence, they will be considered separately.

We introduce our first classifying index $\mu = 0, 1$ distinguishing these two cases.

Definition 17. We are given a boundary algebra $\mathcal{P}_N^m$ generated by $m \geq 1$ parafermions. We define the classifying index $\mu = 0, 1$, such that

$$\mu = \begin{cases} 0, & \text{if the number of boundary parafermions } m \text{ is even,} \\ 1, & \text{if the number of boundary parafermions } m \text{ is odd.} \end{cases} \quad (5.31)$$

5.3.1 Case m -even, i.e., $\mu = 0$

Strategy

We first show that, in the $\mu = 0$ case, any boundary representation is projectively equivalent to a *normal* one that factorizes into two independent parts: a component associated with the parafermion parity symmetry $\mathbb{Z}_N^{\text{PF}}$, and a component associated with the internal symmetry group G , whose structure we leave completely general.

Moreover, we show that the representation can be lifted to a linear representation on the parafermion parity sector $\mathbb{Z}_N^{\text{PF}}$. This reflects the fact that the second cohomology group $H^2[\mathbb{Z}_N^{\text{PF}}, \text{U}(1)]$ is trivial.

We then demonstrate that the remaining nontrivial structure of the boundary representation is fully characterized by a pair of topological indices measuring

1. the obstruction to implementing charge conjugation as a linear symmetry, and
2. the projective representation class of the internal symmetry group $G \cong G^{\text{PF}}/\mathbb{Z}_N^{\text{PF}}$.

Normal form of projective representations

We begin by computing some proportionality phase factors for some generic projective boundary representation U . These will be used later on to explicitly define projectively equivalent representations.

Lemma 10 (Projective proportionality factors). Let U be a projective representation as in **Definition 15**. There exists the set of $\text{U}(1)$ -valued phase factors

$$e^{i\alpha_z}, e^{i\alpha_N} \in \text{U}(1), \quad \forall z \in \mathbb{Z}_N, \quad (5.32a)$$

$$e^{i\beta_{(z,g)}} \in \text{U}(1), \quad \forall z \in \mathbb{Z}_N \setminus \{0\}, \quad \forall g \in G \setminus \{e\}, \quad (5.32b)$$

such that

$$U[0, e] = e^{i\alpha_0} \mathbb{1}, \quad (5.32c)$$

$$U[1, e]^z = e^{i\alpha_z} U[z, e] \quad \forall z \in \mathbb{Z}_N, \quad (5.32d)$$

$$U[1, e]^N = e^{i\alpha_N} \mathbb{1}, \quad (5.32e)$$

$$U[z, g] = e^{-i\beta_{(z, g)}} U[z, e] U[0, g], \quad \forall z \in \mathbb{Z}_N \setminus \{0\}, \quad \forall g \in G \setminus \{e\}. \quad (5.32f)$$

Proof. For existence, it suffices to use the additive structure of $\mathbb{Z}_N$ in combination with the group extension structure of G^{PF} to deduce the above identities up to proportionality. The partition from **Lemma 9** guarantees that the expressions are well-defined. $\square$

We use these proportionality phase factors to explicitly lift much of the projectivity of a general representation V .

Proposition 5 (Normal form of projective representations for even m). Let $V (V')$ be a projective representation as in **Definition 15**. Then $V (V')$ is always projectively equivalent to a projective representation $U (U')$ such that the following properties hold.

1. The identities

$$U[0, e] = \mathbb{1}, \quad (5.33a)$$

$$U[z, e] = U[1, e]^z, \quad \forall z \in \mathbb{Z}_N, \quad (5.33b)$$

$$U[1, e]^N = \mathbb{1}, \quad (5.33c)$$

$$U[z, g] = U[z, e] U[0, g], \quad \forall z \in \mathbb{Z}_N \setminus \{0\}, \quad \forall g \in G \setminus \{e\}, \quad (5.33d)$$

$$U[0, g] = V[0, g], \quad \forall g \in G \setminus \{e\}, \quad (5.33e)$$

hold.

2. The boundary parafermion charge $(1, e)$ is represented by

$$U[1, e] = U'[1, e] \equiv U_{\text{pf}} = \prod_{j=1}^{m/2} \omega_N^{S_N} \gamma_{2j-1} \gamma_{2j}^\dagger, \quad (5.34a)$$

whereby

$$U[z, e] = U'[z, e] = U_{\text{pf}}^z. \quad (5.34b)$$

for all $z \in \mathbb{Z}_N$.

3. The projective representations U and U' differ only due to elements of G ,

$$U[z, g] \neq U'[z, g] \iff U[0, g] \neq U'[0, g]. \quad (5.34c)$$

4. The pair U and U' of normal projective representations are projectively equivalent if and only if there exists a G -dependent map

$$\varphi : G \longrightarrow \text{U}(1) \quad (5.34d)$$

such that

$$U'[z, g] = e^{i\varphi(g)} U[z, g], \quad \varphi(e) = 0, \quad (5.34e)$$

for all $(z, g) \in \mathbb{Z}_N \times G$.

Remark 26. Equation (5.33e) reflects the fact that we do not have to modify the representations of elements of the form $(0, g)$ for $g \in G \setminus \{e\}$.

Remark 27. We have effectively “trivialized” any projective representation of $\mathbb{Z}_N^{\text{PF}} \trianglelefteq G^{\text{PF}}$ on the boundary by showing that we may always lift it to one where

1. the generator of parafermion parity symmetry $(1, e) \in \mathbb{Z}_N \times G$ is the standard parity operator U_{pf} of the algebra generated by an even number of parafermions $\mathcal{P}_N^m$,
2. the representation of the normal symmetry subgroup $\mathbb{Z}_N^{\text{PF}}$ is linear,
3. the representations of symmetries $g \in G \cong G^{\text{PF}}/\mathbb{Z}_N^{\text{PF}}$ are independent of parafermion parity in the sense that $U[z, g] = U[z, e]U[e, g]$ factorizes without projective factors.

Definition 18 (Normal projective representation for even m). **Proposition 5** implies that, when m is even, we may always and without loss of generality consider a projective representation U to have the properties (5.33a)-(5.33d) and represent the boundary parafermion charge operator by U_{pf} as defined in Eq. (5.16). We say that the projective representation U is in its *normal form*.

Proof of Proposition 5. Let $\alpha_z, \beta_{(z,g)} \in \mathbb{R}/2\pi\mathbb{Z}$ be as in **Lemma 10** applied to the projective representation V . We construct a map

$$\theta : (\mathbb{Z}_N \times G) \longrightarrow \mathbb{R}/2\pi\mathbb{Z} \quad (5.35a)$$

by defining it piecewise on the partition given by **Lemma 9**, thereby guaranteeing that it is well-defined.

- For elements in $\{(0, e), (1, e)\}$

$$\theta[0, e] := -\alpha_0, \quad (5.35b)$$

$$\theta[1, e] := -\frac{\alpha_N}{N}, \quad (5.35c)$$

- in $\{(z, e) \mid z \in \mathbb{Z}_N \setminus \{0, 1\}\}$

$$\theta[z, e] := -\alpha_z + \theta[1, e] \cdot z, \quad (5.35d)$$

- in $\{(0, g) \mid g \in G \setminus \{e\}\}$

$$\theta[0, g] = 0, \quad (5.35e)$$

- and in $\{(z, g) \mid z \in \mathbb{Z}_N \setminus \{0\}, g \in G \setminus \{e\}\}$

$$\theta[z, g] := -\beta_{(z,g)} + \theta[z, e]. \quad (5.35f)$$

Note that $\theta[1, e] = \alpha_N/N$ is well-defined as all roots exist in $U(1)$ (this is not so in $\mathbb{Z}_N$).

We define the projectively equivalent representation $U \sim V$

$$U[z, g] := e^{i\theta[z,g]} V[z, g]. \quad (5.36)$$

One computes for U using Eqs. (5.32c)-(5.32f):

- For the identity element,

$$\begin{aligned}
U[0, e] &= e^{i\theta[0, e]} V[0, e] \\
&= e^{-i\alpha_0} e^{i\alpha_0} \mathbb{1} \\
&= \mathbb{1}.
\end{aligned} \tag{5.37a}$$

- For elements in $\{(z, e) \mid z \in \mathbb{Z}_N \setminus \{0\}\}$

$$\begin{aligned}
U[z, e] &= e^{i\theta[z, e]} V[z, e] \\
&= e^{i(-\alpha_z + \theta[1, e] \cdot z)} e^{i\alpha_z} V[1, e]^z \\
&= (e^{i\theta[1, e]} V[1, e])^z \\
&= U[1, e]^z.
\end{aligned} \tag{5.37b}$$

- For the generator of parity symmetry $(1, e)$,

$$\begin{aligned}
U[1, e]^N &= e^{i\theta[1, e] \cdot N} V[1, e]^N \\
&= e^{-i\frac{\alpha_N}{N} \cdot N} e^{i\alpha_N} \mathbb{1} \\
&= \mathbb{1}.
\end{aligned} \tag{5.37c}$$

- For elements in $\{(z, g) \mid z \in \mathbb{Z}_N \setminus \{0\}, g \in G \setminus \{e\}\}$,

$$\begin{aligned}
U[z, g] &= e^{i\theta[z, g]} V[z, g] \\
&= e^{i(-\beta_{(z, g)} + \theta[z, e])} e^{i\beta_{(z, g)}} V[z, e] V[0, g] \\
&= U[z, e] U[0, g],
\end{aligned} \tag{5.37d}$$

where we used the fact that since $\theta[0, g] = 0$ for $g \neq e$,

$$U[0, g] = e^{i\theta[0, g]} V[0, g] = V[0, g]. \tag{5.37e}$$

We move to proving the remaining claims of the proposition.

- Due to the bulk-boundary matching condition, Eq. (5.21), and the fact that the boundary parafermion have global parity one, Eq. (5.13), we have

$$U[1, e] \gamma_i U[1, e]^\dagger \stackrel{!}{=} \tilde{U}[1, e] \gamma_i \tilde{U}[1, e]^\dagger = \omega_N \gamma_i. \tag{5.37f}$$

for all $i = 1, \dots, m$. The same holds for U' . **Lemma 3** states that when m is even, then there exists a up to phase factors unique parafermion parity operator U_{pf} on $\mathcal{P}_N^m$. The ambiguous phase factor can be fixed by modifying θ , whence one find without loss of generality

$$U[1, e] = U'[1, e] = U_{\text{pf}}. \tag{5.37g}$$

- One computes using Eqs. (5.33d) and (5.37f)

$$\begin{aligned}
U[z, g] - U'[z, g] &= U[z, e]U[0, g] - U'[z, e]U'[0, g] \\
&= U[z, e]U[0, g] - U[z, e]U'[0, g] \\
&= U[z, e](U[0, g] - U'[0, g]),
\end{aligned} \tag{5.37h}$$

which vanishes by unitarity of U if and only if $U[0, g] = U'[0, g]$.

- If $V' \sim V$, then there exists a map $\tilde{\varphi} : G \rightarrow \text{U}(1)$ such that

$$V'[0, g] = e^{i\tilde{\varphi}(g)} V[0, g] \tag{5.37i}$$

for all $g \in G$. If θ and θ' are the maps respectively yielding the projectively equivalent representations $U \sim V$ and $U' \sim V'$, then

$$\begin{aligned}
U'[0, g] &= e^{-i\theta'[0, g]} V'[0, g] \\
&= e^{i(\tilde{\varphi}(g) - \theta'[0, g])} V[0, g] \\
&= e^{i(\tilde{\varphi}(g) - \theta'[0, g] + \theta[0, g])} U[0, g] \\
&\equiv e^{i\varphi(g)} U[0, g],
\end{aligned} \tag{5.37j}$$

where $\varphi(g) := \tilde{\varphi}(g) - \theta'[0, g] + \theta[0, g]$. Moreover, using Eqs. (5.33d) and (5.37f) again yields

$$\begin{aligned}
U'[z, g] &= U'[z, e] U'[0, g] \\
&= e^{i\varphi(g)} U'[z, e] U[0, g] \\
&= e^{i\varphi(g)} U[z, e] U[0, g] \\
&= e^{i\varphi(g)} U[z, g].
\end{aligned} \tag{5.37k}$$

The necessity of the statement follows due to the transitivity of the projective equivalence relation ‘ $\sim$ ’.

□

Parafermion charge-conjugation index κ

We introduce our first classifying index κ . When G^{PF} is a central extensions, i.e., $\mathbb{Z}_N^{\text{PF}} \subset \mathcal{Z}(G^{\text{PF}})$ commutes with the entire group, κ measures the failure of the boundary parafermion charge operator to commute with the symmetry representations. In general, κ measures the projectivity of conjugating the boundary parafermion charge by the boundary representation of the symmetry $g \in G$.

We state the definition, existence, and properties of κ in the following proposition.

Proposition 6 (Classifying index κ for even m). Let U be a projective representation of the protecting symmetries $G^{\text{PF}} \cong (\mathbb{Z}_N \times G^{\text{PF}}, \cdot)$. Without loss of generality, we may assume that U is in the normal form described in **Proposition 5**. Then, there exists a map

$$\kappa : G \longrightarrow \mathbb{Z}_N \tag{5.38a}$$

that is implicitly defined by demanding that

$$U[0, g] U[1, e] U[0, g]^\dagger = \omega_N^{\kappa(g)} U[1, e]^{\rho_g(1)} \quad (5.38b)$$

for all $g \in G$. Moreover, the map κ has the following properties.

1. The map κ is explicitly defined by

$$\omega_N^{\kappa(g)} \equiv U[0, g] U[1, e] U[0, g]^\dagger U[1, e]^{-\rho_g(1)} \quad (5.39a)$$

for all $g \in G$.

2. For all $z \in \mathbb{Z}_N$, $g \in G$

$$U[0, g] U[z, e] U[0, g]^\dagger = \omega_N^{\kappa(g) \cdot z} U[z, e] \quad (5.39b)$$

or, equivalently,

$$U[0, g] U[z, e] = \omega_N^{\kappa(g) \cdot z} U[z, e] U[0, g]. \quad (5.39c)$$

3. The map κ is a twisted 1-cocycle², that is

$$\kappa(g_1) \cdot \rho_{g_2}(1) - \kappa(g_1 g_2) + \kappa(g_2) = 0 \quad (5.39d)$$

for all $g_1, g_2 \in G$.

4. κ is an invariant of projective equivalence classes. In particular, if G^{PF} is a central extension ($\rho_g(1) = 1$ for all $g \in G$), then $\kappa \in Z^1[G, \mathbb{Z}_N]$ is a regular 1-cocycle.

Proof. We begin by showing that a map κ as in Eq. (5.38) exists and is well-defined. According to Eq. (5.7b) from **Lemma 9**,

$$\begin{aligned} (0, g) \cdot (z, e) \cdot (0, g)^{-1} &= (\rho_g(z), e) \\ &= (1, e)^{\rho_g(z)}, \quad z \in \mathbb{Z}_N, g \in G, \end{aligned} \quad (5.40)$$

which is equivalent to

$$(0, g) \cdot (z, e) = (1, e)^{\rho_g(1)} \cdot ((0, g), \quad z \in \mathbb{Z}_N, g \in G. \quad (5.41)$$

Take now $z = 1$, yielding $\rho_g(1) = 1$ since ρ is a group action (recall that $(1, e)$ is the generator of parity symmetry). By Eq. (5.41), there exists for all $g \in G$ a proportionality phase factor $\tilde{\kappa}(g) \in \text{U}(1)$ such that

$$U[0, g] U[1, e] = e^{i\tilde{\kappa}(g)} U[1, e]^{\rho_g(1)} U[0, g]. \quad (5.42)$$

Moreover, since U was chosen to be normal, we have by Eq. (5.33c) $U[1, e]^N = \mathbb{1}$. We thus compute

$$\begin{aligned} U[0, g] &= U[0, g] U[1, e]^N \\ &= (e^{i\tilde{\kappa}(g)})^N U[1, e]^N U[0, g] \\ &= e^{i\tilde{\kappa}(g) \cdot N} U[0, g], \end{aligned} \quad (5.43a)$$

²See **Appendix B** and *Remark 38* for terminology.

which implies $e^{i\tilde{\kappa}(g) \cdot N} = 1$. Hence, there exists $\kappa(g) \in \mathbb{Z}_N$ with

$$\omega_N^{\kappa(g)} \equiv e^{i\tilde{\kappa}(g)}. \quad (5.43b)$$

Equation (5.38) follows by multiplying Eq. (5.42) by $U[0, g]^\dagger$ from the right.

The next two statements are verified by observing that

1. U is unitary, whence we may reshuffle Eq. (5.38) as needed,
2. and U was chosen such that $U[z, e] = U[1, e]^z$, see Eq. (5.33b).

We show that κ satisfies the twisted 1-cocycle condition

$$\kappa(g_1) \cdot \rho_{g_2}(1) - \kappa(g_1 g_2) + \kappa(g_2) = 0 \quad (5.44)$$

for all $g_1, g_2 \in G$.

On the one hand, under conjugation by

$$\begin{aligned} U[0, g_1] U[0, g_2] &= e^{i\varepsilon} U[c(g_1, g_2), g_1 g_2] \\ &= e^{i\varepsilon} U[1, e]^{+c(g_1, g_2)} U[0, g_1 g_2], \quad \varepsilon \in \mathbb{U}(1), \end{aligned} \quad (5.45)$$

the element $U[1, e]$ is mapped to

$$\begin{aligned} U[1, e] &\mapsto \left(U[0, g_1] U[0, g_2] \right) U[1, e] \left(U[0, g_1] U[0, g_2] \right)^\dagger \\ &= \left(U[1, e]^{+c(g_1, g_2)} U[0, g_1 g_2] \right) U[1, e] \left(U[1, e]^{+c(g_1, g_2)} U[0, g_1 g_2] \right)^\dagger \\ &= U[1, e]^{+c(g_1, g_2)} \left(U[0, g_1 g_2] U[1, e] U[0, g_1 g_2]^\dagger \right) U[1, e]^{-c(g_1, g_2)} \\ &= U[1, e]^{+c(g_1, g_2)} \left(\omega_N^{\kappa(g_1 g_2)} U[1, e]^{\rho_{g_1 g_2}(1)} \right) U[1, e]^{-c(g_1, g_2)} \\ &= \omega_N^{\kappa(g_1 g_2)} U[1, e]^{\rho_{g_1 g_2}(1)}, \end{aligned} \quad (5.46)$$

where we made use of the pair of identities

$$U[0, g] U[1, e] U[0, g]^\dagger = \omega_N^{\kappa(g)} U[\rho_g(1), e] \quad (5.47a)$$

and

$$U[\rho_g(1), e] = U[1, e]^{\rho_g(1)} \quad (5.47b)$$

to reach the penultimate equality.

On the other hand, under conjugation by Eq. (5.45), associativity dictates that the element

$U[1, e]$ is mapped to

$$\begin{aligned}
U[1, e] &\mapsto U[0, g_1] \left(U[0, g_2] U[1, e] U[0, g_2]^\dagger \right) U[0, g_1]^\dagger \\
&\stackrel{\text{Eq. (5.47a)}}{=} U[0, g_1] \left(\omega_N^{\kappa(g_2)} U[\rho_{g_2}(1), e] \right) U[0, g_1]^\dagger \\
&= \omega_N^{\kappa(g_2)} \left(U[0, g_1] U[\rho_{g_2}(1), e] U[0, g_1]^\dagger \right) \\
&\stackrel{\text{Eq. (5.47a)}}{=} \omega_N^{\kappa(g_2)} \omega_N^{\kappa(g_1) \cdot \rho_{g_2}(1)} U[\rho_{g_1}(\rho_{g_2}(1)), e] \\
&\stackrel{\text{Eq. (5.47b)}}{=} \omega_N^{\kappa(g_2) + \kappa(g_1) \cdot \rho_{g_2}(1)} U[1, e]^{\rho_{g_1}(\rho_{g_2}(1))} \\
&= \omega_N^{\kappa(g_2) + \kappa(g_1) \cdot \rho_{g_2}(1)} U[1, e]^{(\rho_{g_1} \circ \rho_{g_2})(1)} \\
&= \omega_N^{\kappa(g_2) + \kappa(g_1) \cdot \rho_{g_2}(1)} U[1, e]^{\rho_{g_1 g_2}(1)}, \tag{5.48}
\end{aligned}$$

where we used the identity

$$\rho_{g_1} \circ \rho_{g_2} = \rho_{g_1 g_2} \tag{5.49}$$

to reach the last equality.

Since expression (5.46) must coincide with expression (5.48), we conclude that

$$\kappa(g_1 g_2) = \kappa(g_1) \cdot \rho_{g_2}(1) + \kappa(g_2), \tag{5.50}$$

i.e., the twisted 1-cocycle condition (5.44) must hold.

Finally, a pair of projectively equivalent representation $U \sim U'$ differ from one another only by phase factors, which commute with all operators and hence cancel out under conjugation when conjugating the parity representation $U[1, e]$ in Eq. (5.38). $\square$

Projective representation of G index ν

We introduce our second classifying index ν . **Proposition 5** showed that a pair of projective representations V, V' may always be brought into a equivalent normal form U, U' , respectively. In particular, Eqs. (5.34c) and (4) showed that projective (in)equivalence depended only on elements of G . The following proposition shows that there exists a well-defined 2-cochain ν measuring the projectivity of representations of G . In fact, ν is nothing but the restriction of U 's Schur-multiplier to G .

Proposition 7 (Classifying index ν for even m). Let U, U' be projective representations of the protecting symmetries $G^{\text{PF}} \cong (\mathbb{Z}_N \times G, \cdot)$. Without loss of generality, we may assume that they are in the normal form given by **Proposition 5**. Then, there exists a map

$$\nu : G \times G \longrightarrow \text{U}(1) \tag{5.51a}$$

that is implicitly defined by demanding that

$$U[0, g_1] U[0, g_2] = e^{i\nu(g_1, g_2)} U[c(g_1, g_2), g_1 g_2] \tag{5.51b}$$

for all pairs $g_1, g_2 \in G$. Moreover, the function ν has the following properties.

1. The map ν is explicitly defined by

$$e^{i\nu(g_1, g_2)} \equiv U[1, e]^{-c(g_1, g_2)} U[0, g_1] U[0, g_2] U[0, g_1 g_2]^\dagger \quad (5.52a)$$

for all pairs $g_1, g_2 \in G$.

2. If the Schur-multiplier of U is ϕ , then

$$\nu(g_1, g_2) \equiv \phi[(0, g_1), (0, g_2)], \quad (5.52b)$$

for all pairs $g_1, g_2 \in G$.

3. If κ is the 1-cochain associated to the U given by **Proposition 6**, then ν is a modified 2-cocycle³, i.e., for any $g_1, g_2, g_3 \in G$

$$(\delta^2 \nu)(g_1, g_2, g_3) + \frac{2\pi}{N} \kappa(g_1) \cdot c(g_2, g_3) = 0, \quad (5.52c)$$

where, δ^2 is the (regular) 2-coboundary operator

$$(\delta^2 \nu)(g_1, g_2, g_3) := \nu(g_1, g_2) + \nu(g_1 g_2, g_3) - \nu(g_2, g_3) - \nu(g_1, g_2 g_3) \quad (5.52d)$$

from **Appendix B**. In particular, if G^{PF} is a split extension, i.e., $c = 0$, then $\nu \in Z^2[G, \text{U}(1)]$ is a regular 2-cocycle.

4. If $U \sim U'$ are projectively equivalent, then the respective modified 2-cocycles ν, ν' differ by a (regular) 2-coboundary,

$$(\nu - \nu')(g_1, g_2) = (\delta^1 \varphi)(g_1, g_2) := \varphi(g_1) - \varphi(g_1 g_2) + \varphi(g_2), \quad (5.52e)$$

for all pairs $g_1, g_2 \in G$, where $\varphi : G \rightarrow \text{U}(1)$ is a 1-cochain (projectively relating U' to U).

Proof. Let $\phi : (\mathbb{Z}_N \times G)^2 \rightarrow \text{U}(1)$ be U 's Schur-multiplier. Then we may define

$$\nu(g_1, g_2) := \phi[(0, g_1), (0, g_2)] \quad (5.53a)$$

and, by the definition of the Schur-multiplier,

$$\begin{aligned} U[0, g_1] U[0, g_2] &= e^{i\phi[(0, g_1), (0, g_2)]} U[c(g_1, g_2), g_1 g_2] \\ &\equiv e^{i\nu(g_1, g_2)} U[c(g_1, g_2), g_1 g_2]. \end{aligned} \quad (5.53b)$$

for any pair $g_1, g_2 \in G$. This proves Eqs. (5.51b)-(5.52b), whence ν is well-defined.

We show that ν is a modified 2-cocycle. Because of associativity, we must have the identity

$$\left(U[0, g_1] U[0, g_2] \right) U[0, g_3] = U[0, g_1] \left(U[0, g_2] U[0, g_3] \right). \quad (5.54)$$

³See **Appendix B** and *Remark 38* for terminology.

This associativity combined with the identities

$$(\delta^2 \nu)(g_1, g_2, g_3) := \nu(g_2, g_3) + \nu(g_1, g_2 g_3) - \nu(g_1 g_2, g_3) - \nu(g_1, g_2), \quad (5.55a)$$

$$\left((0, g_1) \cdot (0, g_2) \right) \cdot (0, g_3) = (0, g_1) \cdot \left((0, g_2) \cdot (0, g_3) \right), \quad (5.55b)$$

$$\rho_g(z) = \rho_g(1) \cdot z, \quad (5.55c)$$

delivers a modified 2-cocycle condition by the following computation.

On the one hand,

$$\begin{aligned} \left(U[0, g_1] U[0, g_2] \right) U[0, g_3] &= \left(e^{+i\nu(g_1, g_2)} U[1, e]^{c(g_1, g_2)} U[0, g_1 g_2] \right) U[0, g_3] \\ &= e^{+i\nu(g_1, g_2) + i\nu(g_1 g_2, g_3)} \\ &\quad \times U[1, e]^{c(g_1, g_2) + c(g_1 g_2, g_3)} U[0, g_1 g_2 g_3]. \end{aligned} \quad (5.56)$$

On the other hand,

$$\begin{aligned} U[0, g_1] \left(U[0, g_2] U[0, g_3] \right) &= U[0, g_1] \left(e^{+i\nu(g_2, g_3)} U[1, e]^{c(g_2, g_3)} U[0, g_2 g_3] \right) \\ &= e^{+i\nu(g_2, g_3)} \left(U[0, g_1] U[1, e]^{c(g_2, g_3)} \right) U[0, g_2 g_3] \\ &= e^{+i\nu(g_2, g_3)} \left(\omega_N^{\kappa(g_1) \cdot c(g_2, g_3)} U[1, e]^{\rho_{g_1}(1) \cdot c(g_2, g_3)} U[0, g_1] \right) U[0, g_2 g_3] \\ &= e^{+i\nu(g_2, g_3) + i\nu(g_1, g_2 g_3)} \omega_N^{\kappa(g_1) \cdot c(g_2, g_3)} \\ &\quad \times U[1, e]^{\rho_{g_1}(1) \cdot c(g_2, g_3) + c(g_1, g_2 g_3)} U[0, g_1 g_2 g_3]. \end{aligned} \quad (5.57)$$

Now,

$$\begin{aligned} U[1, e]^{c(g_1, g_2) + c(g_1 g_2, g_3)} U[0, g_1 g_2 g_3] &= U[c(g_1, g_2) + c(g_1 g_2, g_3), e] U[0, g_1 g_2 g_3] \\ &= U[c(g_1, g_2) + c(g_1 g_2, g_3), g_1 g_2 g_3] \\ &= U \left[\left((0, g_1) \cdot (0, g_2) \right) \cdot (0, g_3) \right], \end{aligned} \quad (5.58)$$

while

$$\begin{aligned} U[1, e]^{\rho_{g_1}(1) \cdot c(g_2, g_3) + c(g_1, g_2 g_3)} U[0, g_1 g_2 g_3] &= U[\rho_{g_1}(1) \cdot c(g_2, g_3) + c(g_1, g_2 g_3), e] U[0, g_1 g_2 g_3] \\ &= U[\rho_{g_1}(1) \cdot c(g_2, g_3) + c(g_1, g_2 g_3), g_1 g_2 g_3] \\ &= U \left[(0, g_1) \cdot \left((0, g_2) \cdot (0, g_3) \right) \right]. \end{aligned} \quad (5.59)$$

By the associativity (5.55b), it follows that

$$U[1, e]^{c(g_1, g_2) + c(g_1 g_2, g_3)} U[0, g_1 g_2 g_3] = U[1, e]^{\rho_{g_1}(1) \cdot c(g_2, g_3) + c(g_1, g_2 g_3)} U[0, g_1 g_2 g_3]. \quad (5.60)$$

Equating Eqs. (5.56) and (5.57) as is dictated by Eq. (5.54) thus delivers the condition

$$\begin{aligned} 2\pi n &= \nu(g_1, g_2) + \nu(g_1 g_2, g_3) - \nu(g_2, g_3) - \nu(g_1, g_2 g_3) - \frac{2\pi}{N} \kappa(g_1) \cdot c(g_2, g_3) \\ &= -(\delta^2 \nu)(g_1, g_2, g_3) - \frac{2\pi}{N} \kappa(g_1) \cdot c(g_2, g_3) \end{aligned} \quad (5.61)$$

for any integer n .

We show the last statement. Let the pair (U', ν') and the pair (U, ν) be projectively equivalent. By **Proposition 5** this means that there exists a map $\varphi : G \rightarrow U(1)$ such that

$$U'[0, g] = e^{+i\varphi(g)} U[0, g], \quad \forall g \in G, \quad U'[z, e] = U[z, e], \quad \forall z \in \mathbb{Z}_N. \quad (5.62)$$

On the one hand,

$$\begin{aligned} U'[0, g_1] U'[0, g_2] &= e^{+i\nu'(g_1, g_2)} U'[c(g_1, g_2), g_1 g_2] \\ &= e^{+i\nu'(g_1, g_2)} U'[c(g_1, g_2), e] U'[0, g_1 g_2] \\ &= e^{+i\nu'(g_1, g_2)} e^{+i\varphi(g_1 g_2)} U[c(g_1, g_2), e] U[0, g_1 g_2] \\ &= e^{+i\nu'(g_1, g_2)} e^{+i\varphi(g_1 g_2)} U[c(g_1, g_2), g_1 g_2]. \end{aligned} \quad (5.63)$$

On the other hand,

$$\begin{aligned} U'[0, g_1] U'[0, g_2] &= e^{+i[\varphi(g_1) + \varphi(g_2)]} U[0, g_1] U[0, g_2] \\ &= e^{+i[\varphi(g_1) + \varphi(g_2)]} e^{+i\nu(g_1, g_2)} U[c(g_1, g_2), g_1 g_2]. \end{aligned} \quad (5.64)$$

It follows that

$$\nu'(g_1, g_2) + \varphi(g_1 g_2) = \varphi(g_1) + \varphi(g_2) + \nu(g_1, g_2), \quad (5.65)$$

i.e.,

$$\nu'(g_1, g_2) - \nu(g_1, g_2) = \varphi(g_1) - \varphi(g_1 g_2) + \varphi(g_2) \equiv (\delta^1 \varphi)(g_1, g_2). \quad (5.66)$$

□

Schur-Multiplier Decomposition

Having established that the modified 1- and 2-cocycles κ and ν are well-defined, we are now in a position to show that they do indeed classify the boundary projective representations of the protecting symmetry group G^{PF} . We do this by showing that the (regular) 2-cocycle $\phi \in Z^2[\mathbb{Z}_N \times G, U(1)]$ characterizing a projective representation U (as defined in **Subsection 5.2.4**) can be decomposed in terms of κ and ν .

By **Proposition 5**, we may assume U to be normal, thereby fulfilling Eqs. (5.33a)-(5.33d). Let κ and ν be defined for U as in **Propositions 6** and **7** respectively. Consider an arbitrary pair $(z_1, g_1), (z_2, g_2) \in \mathbb{Z}_N \times G$. By definition,

$$U[z_1, g_1] U[z_2, g_2] = e^{i\phi[(z_1, g_1), (z_2, g_2)]} U[(z_1, g_1) \cdot (z_2, g_2)], \quad (5.67a)$$

where ϕ is U 's Schur-multiplier. We then do the manipulations

$$\begin{aligned}
U[z_1, g_1] U[z_2, g_2] &= \left(U[(z_1, e)] U[(0, g_1)] \right) \left(U[(z_2, e)] U[(0, g_2)] \right) \\
&\stackrel{\text{associativity}}{=} U[(z_1, e)] \left(U[(0, g_1)] U[(z_2, e)] \right) U[(0, g_2)] \\
&\stackrel{\text{exchange in parenthesis}}{=} e^{i\kappa(g_1) \cdot z_2} \left(U[(z_1, e)] U[(\rho_{g_1}(z_2), e)] \right) \left(U[(0, g_1)] U[(0, g_2)] \right) \\
&\stackrel{\text{projective representations}}{=} e^{i\kappa(g_1) \cdot z_2} \left(U[(z_1 + \rho_{g_1}(z_2), e)] \right) \left(e^{i\nu(g_1, g_2)} U[(c(g_1, g_2), g_1 g_2)] \right) \\
&\stackrel{\text{factorization}}{=} e^{i\kappa(g_1) \cdot z_2} e^{i\nu(g_1, g_2)} U[(z_1 + \rho_{g_1}(z_2), e)] \left(U[(c(g_1, g_2), e)] U[(0, g_1 g_2)] \right) \\
&\stackrel{\text{associativity}}{=} e^{i\kappa(g_1) \cdot z_2} e^{i\nu(g_1, g_2)} \left(U[(z_1 + \rho_{g_1}(z_2), e)] U[(c(g_1, g_2), e)] \right) U[(0, g_1 g_2)] \\
&= e^{i\kappa(g_1) \cdot z_2} e^{i\nu(g_1, g_2)} \left(U[(z_1 + \rho_{g_1}(z_2) + c(g_1, g_2), e)] U[(0, g_1 g_2)] \right) \\
&= e^{i\kappa(g_1) \cdot z_2} e^{i\nu(g_1, g_2)} U[(z_1 + \rho_{g_1}(z_2) + c(g_1, g_2), g_1 g_2)] \\
&= e^{i\kappa(g_1) \cdot z_2} e^{i\nu(g_1, g_2)} U[(z_1, g_1) \cdot (z_2, g_2)]. \tag{5.67b}
\end{aligned}$$

Since the right-hand sides of Eqs. (5.67a) and (5.67b) must be equal, it follows that

$$\phi[(z_1, g_1), (z_2, g_2)] = \frac{2\pi}{N} \kappa(g_1) \cdot z_2 + \nu(g_1, g_2). \tag{5.68}$$

5.3.2 Case m -odd, i.e., $\mu = 1$

Strategy

We first show that, in the $\mu = 1$ case, any boundary representation is projectively equivalent to a *normal* one that factorizes into two independent parts: a component associated with the parafermion parity symmetry $\mathbb{Z}_N^{\text{PF}}$, and a component associated with the internal symmetry group G , whose structure we leave completely general.

Moreover, we show that the representation can be lifted to a linear representation on the parafermion parity sector $\mathbb{Z}_N^{\text{PF}}$. Contrary to the $\mu = 0$ case, charge conjugation can now also be implemented linearly.

We then demonstrate that the remaining nontrivial structure of the boundary representation is fully characterized by a pair of topological indices measuring

1. the obstruction to implementing conjugation of the center as a linear transformation, and
2. the projective representation class of the internal symmetry group $G \cong G^{\text{PF}} / \mathbb{Z}_N^{\text{PF}}$.

Normal form of projective representations

Let $U : G^{\text{PF}} \rightarrow \mathcal{P}_{N, \infty}^{m+1}$ be a projective boundary representation of the protecting symmetries $G^{\text{PF}} \cong \mathbb{Z}_N \times G$ as in **Definition 16**, i.e., there exists two 2-cochains

$$\phi_1 : (\mathbb{Z}_N \times G)^2 \rightarrow \text{U}(1), \tag{5.69a}$$

$$\phi_2 : (\mathbb{Z}_N \times G)^2 \rightarrow \mathbb{Z}_N, \tag{5.69b}$$

such that

$$U[z_1, g_1] U[z_2, g_2] = e^{i\phi_1[(z_1, g_1), (z_2, g_2)]} Y^{\phi_2[(z_1, g_1), (z_2, g_2)]} U[(z_1, g_1) \cdot (z_2, g_2)] \quad (5.69c)$$

for all pairs $(z_1, g_1), (z_2, g_2) \in \mathbb{Z}_N \times G$.

We begin by showing how the Schur-multiplier ϕ_2 associated to the central factors Y^k can be decomposed into 1-cochains.

Lemma 11 (Decomposition of ϕ_2 in terms of ε for odd m). Given is a projective boundary representation U as in **Definition 16**. Let ϕ_1, ϕ_2 be the Schur-multipliers associated to the $U(1)$ and $\{Y^k\}_{k=0}^{N-1}$ redundancies, respectively. Then,

1. there exists a map

$$\varepsilon : \mathbb{Z}_N \times G \longrightarrow \mathbb{Z}_N \quad (5.70a)$$

such that

$$U[z, g] \gamma_\infty U[z, g]^\dagger = \omega_N^{\varepsilon[z, g]} \gamma_\infty^{\rho_g(1)}, \quad (5.70b)$$

2. and

$$\begin{aligned} \phi_2[(z_1, g_1), (z_2, g_2)] = & \left(\varepsilon[z_1, g_1] \cdot \rho_{g_2}(1) - \varepsilon[(z_1, g_1) \cdot (z_2, g_2)] + \varepsilon[z_2, g_2] \right) \\ & \cdot \rho_{(g_1 g_2)^{-1}}(1) \end{aligned} \quad (5.70c)$$

for any pair $(z_1, g_1), (z_2, g_2) \in \mathbb{Z}_N \times G$.

Remark 28 (Twisted 2-coboundary for odd m). The term in the parenthesis on the right-hand side of equation (5.70c) can be interpreted as a twisted 2-coboundary, as defined in *Remark 38* of **Appendix B**. Thus, **Lemma 11** states that ϕ_2 is itself a twisted 2-coboundary, i.e., it can be gauged away (lifted). We will construct in **Proposition 8** a gauge in which $\varepsilon = 0$ so that $\phi_2 = 0$.

Proof. We begin by showing that ε is well-defined. As assumed in our setup, see Eq. (5.29b), we may define a map

$$\begin{aligned} \tilde{\varepsilon} : \mathbb{Z}_N \times G &\rightarrow \mathbb{R}/2\pi\mathbb{Z}, \\ (z, g) &\mapsto \tilde{\varepsilon}[z, g], \end{aligned} \quad (5.71)$$

such that

$$U[z, g] \gamma_\infty U[z, g]^\dagger = e^{i\tilde{\varepsilon}[z, g]} \gamma_\infty^{\rho_g(1)}. \quad (5.72)$$

Now, if we insert

$$\mathbb{1} = \gamma_\infty^N \quad (5.73)$$

into

$$\begin{aligned} \mathbb{1} &= U[z, g] \mathbb{1} U[z, g]^\dagger \\ &= U[z, g] \gamma_\infty^N U[z, g]^\dagger \\ &= (U[z, g] \gamma_\infty U[z, g]^\dagger)^N \\ &= \left(e^{i\tilde{\varepsilon}[z, g]} \gamma_\infty^{\rho_g(1)} \right)^N \\ &= \left(e^{i\tilde{\varepsilon}[z, g]} \right)^N (\gamma_\infty^N)^{\rho_g(1)} \\ &= \left(e^{i\tilde{\varepsilon}[z, g]} \right)^N, \end{aligned} \quad (5.74)$$

we conclude that

$$e^{i\tilde{\varepsilon}[z,g]} = \omega_N^{\varepsilon[z,g]} \quad (5.75a)$$

for some function

$$\begin{aligned} \varepsilon : \mathbb{Z}_N \times G &\rightarrow \mathbb{Z}_N, \\ (z, g) &\mapsto \varepsilon[z, g]. \end{aligned} \quad (5.75b)$$

We move to proving the decomposition of ϕ_2 . First, we define

$$\begin{aligned} (\gamma_\infty)_{g_2} &:= U[z_2, g_2] \gamma_\infty U[z_2, g_2]^\dagger \\ &= \omega_N^{\varepsilon[z_2, g_2]} \gamma_\infty^{\rho_{g_2}(1)}. \end{aligned} \quad (5.76)$$

Iteration yields

$$\begin{aligned} \left((\gamma_\infty)_{g_2} \right)_{g_1} &:= U[z_1, g_1] (\gamma_\infty)_{g_2} U[z_1, g_1]^\dagger \\ &= U[z_1, g_1] (U[z_2, g_2] \gamma_\infty U[z_2, g_2]^\dagger) U[z_1, g_1]^\dagger \\ \stackrel{\text{Eq. (5.76)}}{=} &\omega_N^{\varepsilon[z_2, g_2]} U[z_1, g_1] \gamma_\infty^{\rho_{g_2}(1)} U[z_1, g_1]^\dagger \\ &= \omega_N^{\varepsilon[z_2, g_2]} (U[z_1, g_1] \gamma_\infty U[z_1, g_1]^\dagger)^{\rho_{g_2}(1)} \\ \stackrel{\text{Eq. (5.76)}}{=} &\omega_N^{\varepsilon[z_2, g_2]} \left(\omega_N^{\varepsilon[z_1, g_1]} \gamma_\infty^{\rho_{g_1}(1)} \right)^{\rho_{g_2}(1)} \\ &= \omega_N^{\varepsilon[z_2, g_2] + \varepsilon[z_1, g_1] \cdot \rho_{g_2}(1)} \gamma_\infty^{\rho_{g_1}(1) \cdot \rho_{g_2}(1)} \\ &= \omega_N^{\varepsilon[z_2, g_2] + \varepsilon[z_1, g_1] \cdot \rho_{g_2}(1)} \gamma_\infty^{\rho_{g_1 g_2}(1)}. \end{aligned} \quad (5.77)$$

We now define

$$(\gamma_\infty)_{g_1 g_2} := \left(U[z_1, g_1] U[z_2, g_2] \right) \gamma_\infty \left(U[z_1, g_1] U[z_2, g_2] \right)^\dagger, \quad (5.78a)$$

where

$$U[z_1, g_1] U[z_2, g_2] = e^{i\phi_1[(z_1, g_1), (z_2, g_2)]} Y^{\phi_2[(z_1, g_1), (z_2, g_2)]} U[(z_1, g_1) \cdot (z_2, g_2)] \quad (5.78b)$$

by the definition of U being a projective representation. Thus,

$$\begin{aligned} (\gamma_\infty)_{g_1 g_2} &= \left(U[z_1, g_1] U[z_2, g_2] \right) \gamma_\infty \left(U[z_1, g_1] U[z_2, g_2] \right)^\dagger \\ &= Y^{+\phi_2[(z_1, g_1), (z_2, g_2)]} \left(U[(z_1, g_1) \cdot (z_2, g_2)] \gamma_\infty U[(z_1, g_1) \cdot (z_2, g_2)]^\dagger \right) Y^{-\phi_2[(z_1, g_1), (z_2, g_2)]} \\ \stackrel{\text{Eq. (5.76)}}{=} &Y^{+\phi_2[(z_1, g_1), (z_2, g_2)]} \left(\omega_N^{\varepsilon[(z_1, g_1) \cdot (z_2, g_2)]} \gamma_\infty^{\rho_{g_1 g_2}(1)} \right) Y^{-\phi_2[(z_1, g_1), (z_2, g_2)]} \\ &= \omega_N^{\varepsilon[(z_1, g_1) \cdot (z_2, g_2)]} \left(Y^{+\phi_2[(z_1, g_1), (z_2, g_2)]} \gamma_\infty^{\rho_{g_1 g_2}(1)} Y^{-\phi_2[(z_1, g_1), (z_2, g_2)]} \right). \end{aligned} \quad (5.79)$$

With the help of

$$Y \gamma_\infty = \omega_N \gamma_\infty Y, \quad (5.80)$$

we arrive at

$$(\gamma_\infty)_{g_1 g_2} = \omega_N^{\varepsilon[(z_1, g_1) \cdot (z_2, g_2)] + \phi_2[(z_1, g_1) \cdot (z_2, g_2)] \cdot \rho_{g_1 g_2}(1)} \gamma_\infty^{\rho_{g_1 g_2}(1)}. \quad (5.81)$$

By associativity,

$$\left((\gamma_\infty)_{g_2} \right)_{g_1} = (\gamma_\infty)_{g_1 g_2}, \quad (5.82)$$

which is equivalent to

$$\varepsilon[z_2, g_2] + \varepsilon[z_1, g_1] \cdot \rho_{g_2}(1) = \varepsilon[(z_1, g_1) \cdot (z_2, g_2)] + \phi_2[(z_1, g_1) \cdot (z_2, g_2)] \cdot \rho_{g_1 g_2}(1). \quad (5.83)$$

As the multiplicative inverse modulo N of $\rho_{g_1 g_2}(1) \in \mathbb{Z}/M\mathbb{Z}$ exists due to $\rho_{(\cdot)}$ being a group action, we can also write

$$\phi_2[(z_1, g_1) \cdot (z_2, g_2)] = \left(\varepsilon[z_1, g_1] \cdot \rho_{g_2}(1) - \varepsilon[(z_1, g_1) \cdot (z_2, g_2)] + \varepsilon[z_2, g_2] \right) \cdot [\rho_{g_1 g_2}(1)]^{-1}. \quad (5.84)$$

□

We will now use **Lemma 11** to explicitly lift much a representation U 's projectivity. The following proposition, analogous to **Proposition 5** for the $\mu = 0$ case, shows that we can find for every projective representation V an equivalent representation U with “normal” properties.

Proposition 8 (Normal form of projective representations for odd m). Let $V (V')$ be a projective representations as in **Definition 16**. Then $V (V')$ is always projectively equivalent to a projective representation $U (U')$ such that the following properties hold.

1. The identities

$$U[0, e] = \mathbb{1}, \quad (5.85a)$$

$$U[z, e] = U[1, e]^z, \quad \forall z \in \mathbb{Z}_N, \quad (5.85b)$$

$$U[1, e]^N = \mathbb{1}, \quad (5.85c)$$

$$U[z, g] = U[z, e] U[0, g], \quad \forall z \in \mathbb{Z}_N \setminus \{0\}, \quad \forall g \in G \setminus \{e\}. \quad (5.85d)$$

$$U[0, g] = Y^{k_g} V[0, g], \quad k_g \in \mathbb{Z}_N, \quad \forall g \in G \setminus \{e\}, \quad (5.85e)$$

hold.

2. Conjugation of the projective representation $U[1, e]$ of the boundary parafermion charge by any $U[z, g]$ is linear, i.e.,

$$U[z, g] U[1, e] U[z, g]^\dagger = U[\rho_g(1), e]. \quad (5.85f)$$

3. The Schur-multiplier ϕ_2 associated to the central factor is trivialized (ϕ_2 is a 2-coboundary and can thus be set to zero), i.e.,

$$U[z_1, g_1] U[z_2, g_2] = e^{i\phi_1[(z_1, g_1), (z_2, g_2)]} U[(z_1, g_1) \cdot (z_2, g_2)] \quad (5.86)$$

for all $(z_1, g_1), (z_2, g_2) \in \mathbb{Z}_N \times G$.

4. The parafermion charge $(1, e)$ is represented by

$$U[1, e] = U'[1, e] \equiv \gamma_\infty^\dagger \quad (5.87a)$$

and

$$U[z, e] = U'[z, e] = \gamma_\infty^{-z} \quad (5.87b)$$

for all $z \in \mathbb{Z}_N$.

5. The projective representations U and U' differ only due to elements of G ,

$$U[z, g] \neq U'[z, g] \iff U[0, g] \neq U'[0, g]. \quad (5.87c)$$

6. The pair U and U' are projectively equivalent if and only if there exists a G -dependent map

$$\varphi : G \longrightarrow U(1) \quad (5.87d)$$

such that

$$U'[z, g] = e^{i\varphi(g)} U[z, g], \quad \varphi(e) = 0, \quad (5.87e)$$

for all $(z, g) \in \mathbb{Z}_N \times G$.

Remark 29. Equation (5.85e) reflects the fact that our normalization modifies the representations of elements of the form $(0, g)$ for $g \in G \setminus \{e\}$ only by powers of Y . We do not specify any complex phase for such symmetries when constructing the projective equivalence, whence there remains a $U(1)$ -redundancy for the operators $U[0, g]$.

Remark 30. We have effectively “trivialized” any projective representation of $\mathbb{Z}_N^{\text{PF}} \trianglelefteq G^{\text{PF}}$ on the boundary by showing that we may always lift it to one for which

1. the representation of the normal symmetry subgroup $\mathbb{Z}_N^{\text{PF}}$ is linear,
2. moreover, the normality of $\mathbb{Z}_N^{\text{PF}} \trianglelefteq G^{\text{PF}}$ is expressed linearly by Eq. (5.86),
3. the representations of symmetries $g \in G \cong G^{\text{PF}} / \mathbb{Z}_N^{\text{PF}}$ are independent of parafermion parity in the sense that $U[z, g] = U[z, e]U[e, g]$ factorizes without projective factors.

Remark 31. We take a closer look at equation (5.86).

1. For every $g \in G$ such that $\rho_g(1) = 1$, we may equivalently write

$$U[0, g] U[1, e] = U[1, e] U[0, g], \quad (5.88)$$

i.e., representations of $g \in G$ corresponding to trivial actions commute with the boundary charge operator.

2. If G^{PF} is a central extension ($\rho_{(\cdot)}$ is trivial), one verifies that

$$\begin{aligned} U[1, e] U[z, g] U[1, e]^\dagger &= U[1, e] U[z, e] U[0, g] U[1, e]^\dagger \\ &= U[z, e] U[1, e] U[0, g] U[1, e]^\dagger \\ &= U[z, e] U[0, g] \\ &= U[z, g], \quad \forall (z, g) \in \mathbb{Z}_N \times G, \end{aligned} \quad (5.89a)$$

which is to say that every “normal” representation of a protecting symmetry carries boundary parafermion charge zero.

3. We deduce that conjugation of the boundary charge operator $U[1, e]$ does not yield a sensible classifying index when $\mu = 1$, as it may always be trivialized. This means that we cannot reuse the index κ from **Proposition 6** for the $\mu = 0$ case.

Definition 19 (Normal projective representation for odd m). **Proposition 8** implies that, when m is odd, we may always and without loss of generality consider a projective representation U to have the properties (5.85a)-(5.85d) and realize conjugation of the boundary parafermion charge operator in a linear fashion as in Eq. (5.86). We say that the projective representation U is in its *normal form*.

Proof of Proposition 8. We are going to make use of the disjoint partition (5.7a).

We begin with the subset of $\mathbb{Z}_N \times G$ given by

$$\left\{ (z, e) \mid z \in \mathbb{Z}_N \right\}. \quad (5.90a)$$

Following **Corollary 2**, we assign to any (z, e) with $z \in \mathbb{Z}_N$ the operator

$$V[z, e] = e^{i\alpha_1(z)} Y^{\alpha_2(z)} \gamma_\infty^{-z}. \quad (5.90b)$$

Through conjugation, the unitary operator $V[z, e]$ measures the parafermion charge ω_N^z for all odd- m generators of $\mathcal{P}_N^m$, while it measures the charge $\omega_N^{\alpha_2(z)}$ with $\alpha_2(z) \in \mathbb{Z}_N$ for γ_∞ . The phase $\alpha_1(z) \in \mathbb{R}/2\pi\mathbb{Z}$ encodes the $U(1)$ ambiguity when assigning a phase to a unitary operator. By the same reasoning, $\alpha_2(z) \in \mathbb{Z}_N$ encodes the ambiguity when assigning a parafermion charge to the auxiliary parafermion γ_∞ .

We turn our attention to the set

$$\left\{ (0, g) \mid g \in G \setminus \{e\} \right\}. \quad (5.91a)$$

According to **Lemma 11**, for any $g \in G \setminus \{e\}$,

$$V[0, g] \gamma_\infty V[0, g]^\dagger = \omega_N^{\varepsilon(g)} \gamma_\infty^{\rho_g(1)}, \quad (5.91b)$$

where

$$\varepsilon(g) \equiv \varepsilon[0, g] \in \mathbb{Z}_N. \quad (5.91c)$$

If $\varepsilon(g) = 0$ and $\rho_g(1) = 1$ held for all $g \in G$, Eq. (5.91b) would imply that G and $\mathbb{Z}_N^{\text{PF}}$ commute. Here, Eq. (5.91b) implements the weaker condition that $\mathbb{Z}_N^{\text{PF}}$ is a normal subgroup of G^{PF} .

Finally, we consider the set

$$\left\{ (z, g) \mid z \in \mathbb{Z}_N \setminus \{0\}, g \in G \setminus \{e\} \right\} \quad (5.92a)$$

for which the projective composition law is

$$V[z, e] V[0, g] = e^{i\beta_1[z, g]} Y^{\beta_2[z, g]} V[z, g] \quad (5.92b)$$

for some function β_1 taking values in $\mathbb{R}/2\pi\mathbb{Z}$ and for some function β_2 taking values in $\mathbb{Z}_N$.

With the help of the 1-cochain

$$\begin{aligned} \Theta : \mathbb{Z}_N \times G &\longrightarrow \left\{ e^{i\varphi} \mid \varphi \in \mathbb{R}/2\pi\mathbb{Z} \right\} \times \left\{ Y^k \mid k \in \mathbb{Z}_N \right\} \cong \text{U}(1) \times \mathbb{Z}_N, \\ (z, g) &\longmapsto \Theta[z, g], \end{aligned} \quad (5.93a)$$

where

$$\Theta[z, g] := \begin{cases} e^{-i\alpha_a(z)} Y^{-\alpha_2(z)}, & z \in \mathbb{Z}_N, g = e, \\ Y^{-\varepsilon(g) \cdot \rho_{g^{-1}}(1)}, & z = 0, g = G \setminus \{e\}, \\ \omega_N^{-\varepsilon(g) \cdot \rho_{g^{-1}}(z)} e^{i\beta_1[z, g]} Y^{\beta_2[z, g]} \Theta[0, g] \Theta[z, e], & z \in \mathbb{Z}_N \setminus \{0\}, g \in G \setminus \{e\}, \end{cases} \quad (5.93b)$$

we define the projective representation U as the gauge-transformed of the projective representation V , i.e.,

$$U[z, g] := \Theta[z, g] V[z, g]. \quad (5.93c)$$

Equation (5.85e) thus follows.

For any (z, e) from set (5.92),

$$\begin{aligned} U[z, e] &= \Theta[z, g] V[z, g] \\ \text{Eq. (5.90b)} \quad &= e^{-i\alpha_a(z)} Y^{-\alpha_2(z)} e^{+i\alpha_1(z)} Y^{+\alpha_2(z)} \gamma_\infty^{-z} \\ &= \gamma_\infty^{-z}. \end{aligned} \quad (5.94)$$

In particular, we find that

$$U[0, e] = \mathbb{1}, \quad (5.95a)$$

$$U[1, e] = \gamma_\infty^{-1} = \gamma_\infty^\dagger, \quad (5.95b)$$

$$U[\rho_g(1), e] = \gamma_\infty^{-\rho_g(1)}, \quad (5.95c)$$

$$U[\rho_e(1), e] = \gamma_\infty^{-\rho_e(1)} = \gamma_\infty^{-1} = \gamma_\infty^\dagger. \quad (5.95d)$$

For the last identity, we used

$$\rho_e(1) = 1. \quad (5.96)$$

We have proven Eqs. (5.85a)-(5.85c) and Eqs. (5.87a)-(5.87b).

We turn our attention to the set (5.91a) and start from substituting V by U in the adjoint of Eq. (5.91b), namely

$$\begin{aligned} U[0, g] \gamma_\infty^\dagger U[0, g]^\dagger &= \left(Y^{-\varepsilon(g) \cdot \rho_{g^{-1}}(1)} V[0, g] \right) \gamma_\infty^\dagger \left(V[0, g]^\dagger Y^{+\varepsilon(g) \cdot \rho_{g^{-1}}(1)} \right) \\ &= Y^{-\varepsilon(g) \cdot \rho_{g^{-1}}(1)} \left(V[0, g] \gamma_\infty^\dagger V[0, g]^\dagger \right) Y^{+\varepsilon(g) \cdot \rho_{g^{-1}}(1)} \\ &= Y^{-\varepsilon(g) \cdot \rho_{g^{-1}}(1)} \left(\omega_N^{-\varepsilon(g)} \gamma_\infty^{-\rho_g(1)} \right) Y^{+\varepsilon(g) \cdot \rho_{g^{-1}}(1)} \\ &= \omega_N^{-\varepsilon(g)} \left(Y^{-\varepsilon(g) \cdot \rho_{g^{-1}}(1)} \gamma_\infty^{-\rho_g(1)} Y^{+\varepsilon(g) \cdot \rho_{g^{-1}}(1)} \right) \\ &= \omega_N^{-\varepsilon(g)} \omega_N^{(-1)^2 \cdot \varepsilon(g) \cdot \rho_{g^{-1}}(1) \cdot \rho_g(1)} \gamma_\infty^{-\rho_g(1)} \\ &= \omega_N^{-\varepsilon(g) + \varepsilon(g) \cdot \rho_{g^{-1}}(1)} \gamma_\infty^{-\rho_g(1)} \\ \rho_{g g^{-1}}(1) = \rho_e(1) = 1 \quad &= \omega_N^{-\varepsilon(g) + \varepsilon(g)} \gamma_\infty^{-\rho_g(1)} \\ &= \gamma_\infty^{-\rho_g(1)}. \end{aligned} \quad (5.97a)$$

With the help of Eq. (5.95d),

$$U[0, g] \gamma_\infty^\dagger U[0, g]^\dagger = U[\rho_e(1), e]. \quad (5.97b)$$

We have proven Eq. (5.85f).

Finally, we consider the set (5.92a) for which

$$\begin{aligned} U[z, g] &:= \Theta[z, g] V[z, g] \\ \text{Eq. (5.92b)} &= \Theta[z, g] \left(e^{-i\beta_1[z, g]} Y^{-\beta_2[z, g]} V[z, e] V[0, g] \right) \\ \text{Eq. (5.93b)} &= \left(\omega_N^{-\varepsilon(g) \cdot \rho_{g^{-1}}(z)} \underline{e^{i\beta_1[z, g]}} \underline{Y^{\beta_2[z, g]}} \Theta[0, g] \Theta[z, e] \right) \left(\underline{e^{-i\beta_1[z, g]}} \underline{Y^{-\beta_2[z, g]}} V[z, e] V[0, g] \right) \\ &= \omega_N^{-\varepsilon(g) \cdot \rho_{g^{-1}}(z)} \Theta[0, g] \Theta[z, e] V[z, e] V[0, g], \end{aligned} \quad (5.98)$$

as $\Theta[0, g]$ and $\Theta[z, e]$ commute with Y . To proceed, we make use of the normal form (5.94) for $U[z, e]$ and the definition (5.93b) for $\Theta[0, g]$ to do the manipulation

$$\begin{aligned} \Theta[0, g] \Theta[z, e] V[z, e] &= \Theta[0, g] U[z, e] \\ &= Y^{-\varepsilon(g) \cdot \rho_{g^{-1}}(1)} \gamma_\infty^{-z} \\ &= \left(Y^{-\varepsilon(g) \cdot \rho_{g^{-1}}(1)} \gamma_\infty^{-z} Y^{+\varepsilon(g) \cdot \rho_{g^{-1}}(1)} \right) Y^{-\varepsilon(g) \cdot \rho_{g^{-1}}(1)} \\ &= \omega_N^{(-1)^2 \cdot \varepsilon(g) \cdot \rho_{g^{-1}}(1) z} \gamma_\infty^{-z} Y^{-\varepsilon(g) \cdot \rho_{g^{-1}}(1)} \\ &\stackrel{\rho_{g^{-1}}(1) z = \rho_{g^{-1}}(z)}{=} \omega_N^{+\varepsilon(g) \cdot \rho_{g^{-1}}(z)} U[z, e] \Theta[0, g]. \end{aligned} \quad (5.99)$$

Insertion of the right-hand side of Eq. (5.99) into the right-hand side of Eq. (5.98) gives

$$\begin{aligned} U[z, g] &= \omega_N^{-\varepsilon(g) \cdot \rho_{g^{-1}}(z)} \omega_N^{+\varepsilon(g) \cdot \rho_{g^{-1}}(z)} U[z, e] \Theta[0, g] V[0, g] \\ &= U[z, e] U[0, g]. \end{aligned} \quad (5.100)$$

We have proven Eq. (5.85d).

According to **Lemma 11**,

$$V[z, g] \gamma_\infty V[z, g]^\dagger = \omega_N^{\varepsilon[z, g]} \gamma_\infty^{\rho_g(1)}, \quad \forall (z, g) \in \mathbb{Z}_N \times G, \quad (5.101)$$

defines implicitly the $\mathbb{Z}_N$ -valued function ε . We are going to show that

$$\varepsilon[z, g] = 0, \quad \forall (z, g) \in \mathbb{Z}_N \times G, \quad (5.102)$$

in the gauge (5.93). Consequently, with the help of **Lemma 11**, the Schur multiplier ϕ_2 is

seen to vanish identically so that Eq. (5.86) holds. Indeed, with the help of Eq. (5.100),

$$\begin{aligned}
U[z, g] \gamma_\infty U[z, g]^\dagger &= \left(U[z, e] U[0, g] \right) \gamma_\infty \left(U[0, g]^\dagger U[z, e]^\dagger \right) \\
&= U[z, e] \left(U[0, g] \gamma_\infty U[0, g]^\dagger \right) U[z, e]^\dagger \\
&= \gamma_\infty^{-z} \left(U[0, g] \left(\gamma_\infty^\dagger \right)^{N-1} U[0, g]^\dagger \right) \gamma_\infty^{+z} \\
&= \gamma_\infty^{-z} \left(U[0, g] \gamma_\infty^\dagger U[0, g]^\dagger \right)^{N-1} \gamma_\infty^{+z} \\
&\stackrel{\text{Eq. (5.95b)}}{=} \gamma_\infty^{-z} \left(U[0, g] U[1, e] U[0, g]^\dagger \right)^{N-1} \gamma_\infty^{+z} \\
&\stackrel{\text{Eq. (5.97b)}}{=} \gamma_\infty^{-z} \left(U[\rho_g(1), e] \right)^{N-1} \gamma_\infty^{+z} \\
&\stackrel{\text{Eq. (5.97a)}}{=} \gamma_\infty^{-z} \gamma_\infty^{-(N-1) \cdot \rho_g(1)} \gamma_\infty^{+z} \\
&= \gamma_\infty^{\rho_g(1)}, \quad \forall (z, g) \in \mathbb{Z}_N \times G.
\end{aligned} \tag{5.103}$$

We continue with the proof of Eq. (5.87c). If U and U' are both of normal form, then

$$\begin{aligned}
0 &= U[z, g] - U'[z, g] \\
&\stackrel{\text{Eq. (5.100)}}{=} U[z, e] U[0, g] - U'[z, e] U'[0, g] \\
&\stackrel{\text{Eq. (5.94)}}{=} \gamma_\infty^{-z} \left(U[0, g] - U'[0, g] \right)
\end{aligned} \tag{5.104a}$$

holds for any $(z, g) \in \mathbb{Z}_N \times G$ if and only if

$$0 = U[0, g] - U'[0, g]. \tag{5.104b}$$

Finally, we show that the pair U and U' of normal projective representations are projectively equivalent if and only if there exists a map $\varphi : G \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ such that

$$U'[z, g] = e^{i\varphi(g)} U[z, g] \tag{5.105a}$$

with

$$\varphi(e) = 0. \tag{5.105b}$$

($\Leftarrow$) If U is a normal projective representation of G^{PF} , so does $e^{i\varphi(g)} U[z, g]$ under the condition that $\varphi(e) = 0$.

($\Rightarrow$) If U and U' are projectively equivalent, then there exists the pair of functions $\varphi_1 : \mathbb{Z}_N \times G \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ and $\varphi_2 : \mathbb{Z}_N \times G \rightarrow \mathbb{Z}_N$ such that

$$U'[z, g] = e^{i\varphi_1[z, g]} Y^{\varphi_2[z, g]} U[z, g] \tag{5.106}$$

for all $(z, g) \in \mathbb{Z}_N \times G$. Because U and U' are normal projective representations [recall Eq. (5.94)],

$$U'[z, e] = U[z, e] = \gamma_\infty^{-z} \tag{5.107}$$

for all $z \in \mathbb{Z}_N$. By Eq. (5.103),

$$\gamma_\infty^{-\rho_g(1)} = U[z, g] \gamma_\infty^\dagger U[z, g]^\dagger = U'[z, g] \gamma_\infty^\dagger U'[z, g]^\dagger \quad \forall (z, g) \in \mathbb{Z}_N \times G. \tag{5.108}$$

For any $(z, g) \in \mathbb{Z}_N \times G$,

$$\begin{aligned}
\gamma_\infty^{-\rho_g(1)} &= U'[z, g] \gamma_\infty^\dagger U'[z, g]^\dagger \\
&= (e^{i\varphi_1[z, g]} Y^{\varphi_2[z, g]} U[z, g]) \gamma_\infty^\dagger (U[z, g]^\dagger Y^{-\varphi_2[z, g]} e^{-i\varphi_1[z, g]}) \\
&= Y^{\varphi_2[z, g]} (U[z, g] \gamma_\infty^\dagger U[z, g]^\dagger) Y^{-\varphi_2[z, g]} \\
&= Y^{\varphi_2[z, g]} \gamma_\infty^{-\rho_g(1)} Y^{-\varphi_2[z, g]}.
\end{aligned} \tag{5.109}$$

By making use of **Corollary 2**, we deduce that

$$\gamma_\infty^{-\rho_g(1)} = \omega_N^{-\varphi_2[z, g] \cdot \rho_g(1)} \gamma_\infty^{-\rho_g(1)}. \tag{5.110}$$

Multiplication of this equation by $\gamma_\infty^{+\rho_g(1)}$ gives the condition

$$1 = \omega_N^{-\varphi_2[z, g] \cdot \rho_g(1)}. \tag{5.111}$$

As $\rho_g(1) \in \mathbb{Z}_N \setminus \{0\}$, we conclude that

$$\varphi_2[z, g] = 0 \bmod N \tag{5.112}$$

so that Eq. (5.106) simplifies to

$$U'[z, g] = e^{i\varphi_1[z, g]} U[z, g]. \tag{5.113}$$

To show that $\varphi_1[z, g]$ is independent of z in Eq. (5.113), we invoke again Eq. (5.94) through the manipulations

$$\begin{aligned}
\gamma_\infty^{-z} U'[0, g] &= U'[z, e] U'[0, g] \\
&= U'[z, g] \\
&= e^{i\varphi_1[z, g]} U[z, g] \\
&= e^{i\varphi_1[z, g]} U[z, e] U[0, g] \\
&= e^{i\varphi_1[z, g]} \gamma_\infty^{-z} U[0, g],
\end{aligned} \tag{5.114}$$

i.e.,

$$U'[0, g] = e^{i\varphi_1[z, g]} U[0, g]. \tag{5.115}$$

Equations (5.115), (5.113), and (5.107) can only hold simultaneously for all $(z, g) \in \mathbb{Z}_N \times G$ if

$$\varphi_1[z, g] = \varphi_1[0, g] \equiv \varphi_1(g), \quad \varphi_1(e) = 0. \tag{5.116}$$

□

Center generator conjugation index κ

We use the normal form of projective representations to introduce our first classifying index κ .

In the $\mu = 0$ case, the first classifying index was defined in **Proposition 6** by conjugating the boundary representation of parafermion charge. This was natural, as by **Assumption 2**, $\mathbb{Z}_N^{\text{PF}}$ is a normal subgroup of the protecting symmetries. However, as was shown in **Proposition 8**, when $\mu = 1$, one can always lift the representation so that the conjugation of parafermion charge is linear, thereby trivializing such a charge conjugation index.

We will instead conjugate the generator of the boundary algebra's center Y as defined in Eq. (5.23). Recall that this operator was intrinsic to the odd- m parafermion algebra. We state the definition, existence, and properties of such an index κ in the following proposition.

Proposition 9 (Classifying index κ for odd m). Let U be a projective boundary representation of the protecting symmetries $G^{\text{PF}} \cong (\mathbb{Z}_N \times G, \cdot)$ as in **Definition 16**. Without loss of generality, we may assume that U is in the normal form described in **Proposition 8**. Then, there exists a map

$$\kappa : G \longrightarrow \mathbb{Z}_N \quad (5.117a)$$

that is implicitly defined by demanding that

$$U[0, g] Y U[0, g]^\dagger = \omega_N^{\kappa(g)} Y^{\rho_g(1)} \quad (5.117b)$$

for all $g \in G$. Moreover, the function κ has the following properties.

1. The map κ is explicitly defined by

$$\omega_N^{\kappa(g)} \equiv U[0, g] Y U[0, g]^\dagger Y^{-\rho_g(1)} \quad (5.118a)$$

for all $g \in G$.

2. For all $(z, g) \in \mathbb{Z}_N \times G$,

$$U[z, g] Y U[z, g]^\dagger = \omega_N^{\kappa(g) + \rho_g(z)} Y^{\rho_g(1)}, \quad (5.118b)$$

or, equivalently,

$$U[z, g] Y = \omega_N^{\kappa(g) + \rho_g(z)} Y^{\rho_g(1)} U[z, g]. \quad (5.118c)$$

3. The map κ is a modified twisted 1-cocycle⁴, that is

$$\kappa(g_1) \cdot \rho_{g_2}(1) - \kappa(g_1 g_2) + \kappa(g_2) = c(g_1, g_2) \cdot \rho_{g_1 g_2}(1) \quad (5.118d)$$

for all $g_1, g_2 \in G$.

4. The map κ is an invariant of projective equivalence classes. In particular, if G^{PF} is a split extension ($c = 0$), then the map κ is an unmodified twisted 1-cocycle.

⁴See **Appendix B** and *Remark 38* for terminology.

Remark 32 (Interpretations of κ). Consider elements $g \in G$ such that the group action is trivial, i.e., $\rho_g(z) = z$.

- Equation (5.118a) then yields the commutation relation

$$U[0, g] Y = \omega_N^{\kappa(g)} Y U[z, g]. \quad (5.119a)$$

In particular, when G^{PF} is a central extension ($\rho_{(\cdot)}$ is trivial), $\kappa(g)$ encodes the obstruction to a representation of $g \in G$ commuting with the generator Y of the center $\mathcal{Z}(\mathcal{P}_N^m)$.

- Recall that the identity

$$Y \gamma_\infty Y^\dagger = \omega_N \gamma_\infty \quad (5.119b)$$

holds for the auxiliary parafermion γ_∞ . This identity implies that, for any operator $V \in \mathcal{P}_N^m$,

$$Y(V \gamma_\infty^k) Y^\dagger = \omega_N^k (V \gamma_\infty^k) \quad (5.119c)$$

In other words, the adjoint action of the center on operators $U \in \mathcal{P}_{N,\infty}^{m+1}$ that are homogeneous in γ_∞ measures the order of the auxiliary parafermion in U . Rewriting Eq. (5.119a) as

$$Y U[0, g] Y^\dagger = \omega_N^{-\kappa(g)} U[0, g] \quad (5.119d)$$

delivers the interpretation of $\kappa(g)$ as counting the number of times $\gamma_\infty^\dagger$ appears in the boundary projective representation of $g \in G$.

Proof of Proposition 9. By Eq. (5.30b), one has

$$U[0, g] Y U[0, g]^\dagger = e^{i\tilde{\kappa}(g)} Y \rho_g(1) \quad (5.120)$$

for some $\tilde{\kappa}(g) \in \mathbb{R}/2\pi\mathbb{Z}$. We do the manipulations

$$\begin{aligned} \mathbb{1} &= Y^N \\ &= (U[0, g] Y U[0, g]^\dagger)^N \\ &= [e^{i\tilde{\kappa}(g)} Y \rho_g(1)]^N \\ &= [e^{i\tilde{\kappa}(g)}]^N (Y^N)^{\rho_g(1)} \\ &= [e^{i\tilde{\kappa}(g)}]^N. \end{aligned} \quad (5.121)$$

We conclude that

$$e^{i\tilde{\kappa}(g)} = \omega_N^{\kappa(g)}, \quad (5.122)$$

for some map $\kappa : G \rightarrow \mathbb{Z}_N$. Hence, κ exists and we have proven Eq. (5.117).

We prove equation (5.118b). Since U is assumed to fulfill the properties of **Proposition 9**, we compute

$$\begin{aligned} U[z, g] Y U[z, g]^\dagger &= U[z, e] U[0, g] Y U[0, g]^\dagger U[z, e]^\dagger \\ &= \omega_N^{\kappa(g)} \gamma_\infty^{-z} Y \rho_g(1) \gamma_\infty^{+z} \\ &= \omega_N^{\kappa(g)+z \cdot \rho_g(1)} Y \rho_g(1) \\ &= \omega_N^{\kappa(g)+\rho_g(z)} Y \rho_g(1), \end{aligned} \quad (5.123)$$

where we used that Y has parafermion charge one.

Let us now deduce the modified twisted 1-cocycle condition (5.118d) obeyed by κ . We start by defining

$$\begin{aligned} Y_{g_2} &:= U[0, g_2] Y U[0, g_2]^\dagger \\ &= \omega_N^{\kappa(g_2)} Y^{\rho_{g_2}(1)}. \end{aligned} \quad (5.124)$$

We compute by iterating

$$\begin{aligned} (Y_{g_2})_{g_1} &:= U[0, g_1] Y_{g_2} U[0, g_1]^\dagger \\ &= U[0, g_1] (U[0, g_2] Y U[0, g_2]^\dagger) U[0, g_1]^\dagger \\ &\stackrel{\text{Eq. (5.124)}}{=} \omega_N^{\kappa(g_2)} U[0, g_1] Y^{\rho_{g_2}(1)} U[0, g_1]^\dagger \\ &= \omega_N^{\kappa(g_2)} (U[0, g_1] Y U[0, g_1]^\dagger)^{\rho_{g_2}(1)} \\ &= \omega_N^{\kappa(g_2)} (\omega_N^{\kappa(g_1)} Y^{\rho_{g_1}(1)})^{\rho_{g_2}(1)} \\ &= \omega_N^{\kappa(g_2) + \kappa(g_1) \cdot \rho_{g_2}(1)} Y^{\rho_{g_1}(1) \rho_{g_2}(1)} \\ &= \omega_N^{\kappa(g_2) + \kappa(g_1) \cdot \rho_{g_2}(1)} Y^{\rho_{g_1 g_2}(1)}. \end{aligned} \quad (5.125)$$

Analogously, we define

$$Y_{g_1 g_2} := (U[0, g_1] U[0, g_2]) Y (U[0, g_1] U[0, g_2])^\dagger, \quad (5.126a)$$

where

$$U[0, g_1] U[0, g_2] = e^{i\phi[(0, g_1), (0, g_2)]} U[c(g_1, g_2), g_1 g_2]. \quad (5.126b)$$

Thus,

$$\begin{aligned} Y_{g_1 g_2} &= U[c(g_1, g_2), g_1 g_2] Y U[c(g_1, g_2), g_1 g_2]^\dagger \\ &= U[c(g_1, g_2), e] (U[0, g_1 g_2] Y U[0, g_1 g_2]^\dagger) U[c(g_1, g_2), e]^\dagger \\ &= U[c(g_1, g_2), e] [\omega_N^{\kappa(g_1 g_2)} Y^{\rho_{g_1 g_2}(1)}] U[c(g_1, g_2), e]^\dagger \\ &= \omega_N^{\kappa(g_1 g_2)} [\gamma_\infty^{-c(g_1, g_2)} Y^{\rho_{g_1 g_2}(1)} \gamma_\infty^{+c(g_1, g_2)}] \\ &= \omega_N^{\kappa(g_1 g_2) + c(g_1, g_2) \cdot \rho_{g_1 g_2}(1)} Y^{\rho_{g_1 g_2}(1)}. \end{aligned} \quad (5.127)$$

By associativity,

$$(Y_{g_2})_{g_1} = Y_{g_1 g_2}, \quad (5.128)$$

which is equivalent to the twisted 1-coboundary relation

$$\kappa(g_1) \cdot \rho_{g_2}(1) + \kappa(g_2) - \kappa(g_1 g_2) = c(g_1, g_2) \cdot \rho_{g_1 g_2}(1). \quad (5.129)$$

We end with the invariance of κ under projective equivalence. We are given projectively equivalent representations $U \sim U'$ associated to the maps κ and κ' respectively. They are

without loss of generality assumed to be in the normal form given by **Proposition 8**. Hence, there is a map $\varphi : G \rightarrow \text{U}(1)$ such that

$$U'[z, g] := e^{i\varphi(g)} U[z, g], \quad \varphi(g) = 0. \quad (5.130)$$

for all $g \in G$. As κ' and κ are defined by conjugation of Y , and the phase factors $e^{i\varphi(g)}$ commute with all operators, we must have the identity $\kappa' = \kappa$. $\square$

Projective representation of G index ν

We introduce our second classifying index ν , which is analogous to the version presented in **Proposition 7** for the $\mu = 0$ case (m -even). **Proposition 8** showed that a pair of projective representations V, V' may always be brought into a equivalent normal form U, U' , respectively. In particular, Eqs. (5.87c) and (5.87e) showed that projective (in)equivalence depended only on elements of G . The following proposition shows that there exists a well-defined 2-cochain ν measuring the projectivity of representations of G . In fact, ν is nothing but the restriction of U 's Schur-multiplier to G .

Proposition 10 (Classifying index ν for odd m). Let U and U' be projective boundary representations of the protecting symmetries $G^{\text{PF}} \cong (\mathbb{Z}_N \times G, \cdot)$ as given in **Definition 16**. Without loss of generality, we may assume that they are in the normal form given by **Proposition 8**. Then there exists a map

$$\nu : G \times G \longrightarrow \text{U}(1) \quad (5.131a)$$

that is implicitly defined by demanding that

$$U[0, g_1] U[0, g_2] = e^{i\nu(g_1, g_2)} U[c(g_1, g_2), g_1 g_2] \quad (5.131b)$$

for all pairs $g_1, g_2 \in G$. Moreover, the function ν has the following properties.

1. It is explicitly given by

$$e^{i\nu(g_1, g_2)} = U[1, e]^{-c(g_1, g_2)} U[0, g_1] U[0, g_2] U[0, g_1 g_2]^\dagger. \quad (5.132a)$$

2. It is alternatively given by

$$\nu(g_1, g_2) = \phi_1[(0, g_1), (0, g_2)], \quad (5.132b)$$

where ϕ_1 is the Schur-multiplier taking values in $\mathbb{R}/2\pi\mathbb{Z} \cong \text{U}(1)$.

3. It is a (regular) 2-cocycle, i.e.,

$$(\delta^2 \nu)(g_1, g_2, g_3) := \nu(g_2, g_3) - \nu(g_1 g_2, g_3) + \nu(g_1, g_2 g_3) - \nu(g_1, g_2) = 0, \quad (5.132c)$$

where δ^2 is the (regular) 2-coboundary operator.

4. If $U \sim U'$ are projectively equivalent, then their respective 2-cocycles ν and ν' differ by a (regular) 2-coboundary,

$$(\nu - \nu')(g_1, g_2) = (\delta^1 \varphi)(g_1, g_2) := \varphi(g_1) - \varphi(g_1 g_2) + \varphi(g_2), \quad (5.132d)$$

where $\varphi : G \rightarrow \mathbb{U}(1)$ is a 1-cochain (projectively relating U' to U).

5. The equivalence classes of $\{\nu\}$ associated to the set of projective representations $\{U\}$ is classified by the (regular) second group cohomology

$$[\nu] \in H^2[G, \mathbb{U}(1)] \quad (5.132e)$$

as defined in **Appendix B**.

Proof. Existence of a 2-cochain satisfying condition (5.131) Given is a projective representation U in its normal form given by **Proposition 8**, i.e., it obeys the composition rule

$$U[0, g_1] U[0, g_2] = e^{i\phi_1[(0, g_1), (0, g_2)]} U[c(g_1, g_2), g_1 g_2]. \quad (5.133)$$

Define

$$\begin{aligned} \nu : G \times G &\longrightarrow \mathbb{R}/2\pi\mathbb{Z}, \\ (g_1, g_2) &\longmapsto \nu(g_1, g_2) := \phi_1[(0, g_1), (0, g_2)]. \end{aligned} \quad (5.134)$$

We can view ν as the “restriction” of ϕ_1 to $G \times G$. Equations (5.131) and (5.132b) have been proved. Equation (5.132a) is a consequence of multiplying from the right Eq. (5.131b) by $U[c(g_1, g_2), g_1 g_2]^\dagger$.

Proof of Eq. (5.132c) Given are $g_1, g_2, g_3 \in G$. We are going to make extensive use of associativity

$$\left((0, g_1) \cdot (0, g_2) \right) \cdot (0, g_3) = (0, g_1) \cdot \left((0, g_2) \cdot (0, g_3) \right), \quad (5.135a)$$

$$\left(U[0, g_1] U[0, g_2] \right) U[0, g_3] = U[0, g_1] \left(U[0, g_2] U[0, g_3] \right). \quad (5.135b)$$

On the one hand,

$$\begin{aligned} \left((0, g_1) \cdot (0, g_2) \right) \cdot (0, g_3) &= (0 + \rho_{g_1}(0) + c(g_1, g_2), g_1 g_2) \cdot (0, g_3) \\ &\stackrel{\rho_{g_1}(0)=0}{=} (c(g_1, g_2), g_1 g_2) \cdot (0, g_3) \\ &= (c(g_1, g_2) + \rho_{g_1 g_2}(0) + c(g_1 g_2, g_3), g_1 g_2 g_3) \\ &\stackrel{\rho_{g_1 g_2}(0)=0}{=} (c(g_1, g_2) + c(g_1 g_2, g_3), g_1 g_2 g_3). \end{aligned} \quad (5.136a)$$

On the other hand,

$$\begin{aligned} (0, g_1) \cdot \left((0, g_2) \cdot (0, g_3) \right) &= (0, g_1) \cdot (0 + \rho_{g_2}(0) + c(g_2, g_3), g_2 g_3) \\ &\stackrel{\rho_{g_2}(0)=0}{=} (0, g_1) \cdot (c(g_2, g_3), g_2 g_3) \\ &= \left(0 + \rho_{g_1}(c(g_2, g_3)) + c(g_1, g_2 g_3), g_1 g_2 g_3 \right) \\ &= \left(\rho_{g_1}(c(g_2, g_3)) + c(g_1, g_2 g_3), g_1 g_2 g_3 \right). \end{aligned} \quad (5.136b)$$

By associativity of the composition rule $\cdot$,

$$c(g_1, g_2) + c(g_1 g_2, g_3) = \rho_{g_1}(c(g_2, g_3)) + c(g_1, g_2 g_3). \quad (5.136c)$$

On the one hand, with the help of Eq. (5.134),

$$\begin{aligned} & \left(U[0, g_1] U[0, g_2] \right) U[0, g_3] = e^{i\nu(g_1, g_2)} U[c(g_1, g_2), g_1 g_2] U[0, g_3] \\ & \text{factorization} = e^{i\nu(g_1, g_2)} \left(U[c(g_1, g_2), e] U[0, g_1 g_2] \right) U[0, g_3] \\ & \text{associativity} = e^{i\nu(g_1, g_2)} U[c(g_1, g_2), e] \left(U[0, g_1 g_2] U[0, g_3] \right) \\ & \text{Eq. (5.134)} = e^{i\nu(g_1, g_2) + i\nu(g_1 g_2, g_3)} U[c(g_1, g_2), e] U[c(g_1 g_2, g_3), g_1 g_2 g_3] \\ & \text{factorization} = e^{i\nu(g_1, g_2) + i\nu(g_1 g_2, g_3)} U[c(g_1, g_2), e] \left(U[c(g_1 g_2, g_3), e] U[0, g_1 g_2 g_3] \right) \\ & \text{associativity} = e^{i\nu(g_1, g_2) + i\nu(g_1 g_2, g_3)} \left(U[c(g_1, g_2), e] U[c(g_1 g_2, g_3), e] \right) U[0, g_1 g_2 g_3] \\ & \text{projective composition} = e^{i\nu(g_1, g_2) + i\nu(g_1 g_2, g_3)} U[c(g_1, g_2) + c(g_1 g_2, g_3), e] U[0, g_1 g_2 g_3] \\ & \text{projective composition} = e^{i\nu(g_1, g_2) + i\nu(g_1 g_2, g_3)} U[c(g_1, g_2) + c(g_1 g_2, g_3), g_1 g_2 g_3] \\ & \text{Eq. (5.136a)} = e^{i\nu(g_1, g_2) + i\nu(g_1 g_2, g_3)} U[(0, g_1) \cdot (0, g_2) \cdot (0, g_3)]. \end{aligned} \quad (5.137)$$

On the other hand, with the help of Eq. (5.134),

$$\begin{aligned} & U[0, g_1] \left(U[0, g_2] U[0, g_3] \right) = e^{i\nu(g_2, g_3)} U[0, g_1] U[c(g_2, g_3), g_2 g_3] \\ & \text{factorization} = e^{i\nu(g_2, g_3)} U[0, g_1] \left(U[c(g_2, g_3), e] U[0, g_2 g_3] \right) \\ & \text{associativity} = e^{i\nu(g_2, g_3)} \left(U[0, g_1] U[c(g_2, g_3), e] \right) U[0, g_2 g_3] \\ & \text{exchange} = e^{i\nu(g_2, g_3)} \left(U[\rho_{g_1}(c(g_2, g_3)), e] U[0, g_1] \right) U[0, g_2 g_3] \\ & \text{associativity} = e^{i\nu(g_2, g_3)} U[\rho_{g_1}(c(g_2, g_3)), e] \left(U[0, g_1] U[0, g_2 g_3] \right) \\ & \text{Eq. (5.134) with } \rho_{g_1}(0) = 0 = e^{i\nu(g_2, g_3) + i\nu(g_1, g_2 g_3)} U[\rho_{g_1}(c(g_2, g_3)), e] U[c(g_1, g_2 g_3), g_1 g_2 g_3] \\ & \text{factorization} = e^{i\nu(g_2, g_3) + i\nu(g_1, g_2 g_3)} U[\rho_{g_1}(c(g_2, g_3)), e] \left(U[c(g_1, g_2 g_3), e] U[0, g_1 g_2 g_3] \right) \\ & \text{associativity} = e^{i\nu(g_2, g_3) + i\nu(g_1, g_2 g_3)} \left(U[\rho_{g_1}(c(g_2, g_3)), e] U[c(g_1, g_2 g_3), e] \right) U[0, g_1 g_2 g_3] \\ & \text{projective composition} = e^{i\nu(g_2, g_3) + i\nu(g_1, g_2 g_3)} U[\rho_{g_1}(c(g_2, g_3)) + c(g_1, g_2 g_3), e] U[0, g_1 g_2 g_3] \\ & \text{projective composition} = e^{i\nu(g_2, g_3) + i\nu(g_1, g_2 g_3)} U[\rho_{g_1}(c(g_2, g_3)) + c(g_1, g_2 g_3), g_1 g_2 g_3] \\ & \text{Eq. (5.136b)} = e^{i\nu(g_2, g_3) + i\nu(g_1, g_2 g_3)} U[(0, g_1) \cdot (0, g_2) \cdot (0, g_3)]. \end{aligned} \quad (5.138)$$

By the associativity (5.135b) of the operator product, we have found the identity

$$\nu(g_1, g_2) + \nu(g_1 g_2, g_3) = \nu(g_2, g_3) + \nu(g_1, g_2 g_3), \quad (5.139)$$

which we rewrite as the 2-cocycle condition

$$(\delta^2 \nu)(g_1, g_2, g_3) := \nu(g_2, g_3) - \nu(g_1 g_2, g_3) + \nu(g_1, g_2 g_3) - \nu(g_1, g_2) = 0. \quad (5.140)$$

Proof of Eq. (5.132d) We are given the pair U and U' of projective representations in the factorized gauge. By assumption, they are projectively equivalent, i.e., there exists the function

$$\begin{aligned}\theta : G &\longrightarrow \mathbb{R}/2\pi\mathbb{Z}, \\ g &\longmapsto \theta(g),\end{aligned}\tag{5.141}$$

such that

$$U'[0, g] = e^{i\theta(g)} U[0, g]\tag{5.142}$$

for all $g \in G$. We have already shown that we may associate to U (U') the 2-cocycle ν (ν'). Choose any pair $g_1, g_2 \in G$. On the one hand,

$$\begin{aligned}U'[0, g_1] U'[0, g_2] &= e^{i\nu'(g_1, g_2)} U'[c(g_1, g_2), g_1 g_2] \\ &= e^{i\nu'(g_1, g_2) + i\theta(g_1 g_2)} U[c(g_1, g_2), g_1 g_2].\end{aligned}\tag{5.143}$$

On the other hand,

$$\begin{aligned}U'[0, g_1] U'[0, g_2] &= e^{i\theta(g_1) + i\theta(g_2)} U[0, g_1] U[0, g_2] \\ &= e^{i\theta(g_1) + i\theta(g_2) + i\nu(g_1, g_2)} U[c(g_1, g_2), g_1 g_2].\end{aligned}\tag{5.144}$$

It follows that

$$\nu'(g_1, g_2) + \theta(g_1 g_2) = \theta(g_1) + \theta(g_2) + \nu(g_1, g_2),\tag{5.145}$$

which we may rewrite as the 1-coboundary condition

$$\nu'(g_1, g_2) - \nu(g_1, g_2) = \theta(g_1) - \theta(g_1 g_2) + \theta(g_2) \equiv (\delta^1 \theta)(g_1, g_2).\tag{5.146}$$

The last claim follows directly from the previous two by definition of second group cohomology. □

Schur-Multiplier Decomposition

Having established that the modified 1- and 2-cocycles κ and ν are well-defined, we are now in a position to show that they do indeed classify the boundary projective representations of the protecting symmetry group G^{PF} . We do this by showing that the Schur-multipliers $\phi_1 \in C^2[\mathbb{Z}_N \times G, \text{U}(1)]$ and $\phi_2 \in C^2[\mathbb{Z}_N \times G, \text{U}(1)]$ characterizing a projective representation U (as in **Definition 16**) can be decomposed in terms of ν .

By **Proposition 8**, we may assume U to be normal. In particular, the Schur-multiplier associated to the generator of center Y vanishes ϕ_2 . Let ν be defined for U as in **Proposition 10**. Consider an arbitrary pair $(z_1, g_1), (z_2, g_2) \in \mathbb{Z}_N^{\text{PF}} \times G$. By definition,

$$U[z_1, g_1] U[z_2, g_2] = e^{i\phi_1[(z_1, g_1), (z_2, g_2)]} U[(z_1, g_1) \bullet (z_2, g_2)],\tag{5.147}$$

We then do the manipulations

$$\begin{aligned}
U[z_1, g_1] U[z_2, g_2] &= \left\{ U[(z_1, e)] U[(0, g_1)] \right\} \left\{ U[(z_2, e)] U[(0, g_2)] \right\} \\
&\stackrel{\text{associativity}}{=} U[(z_1, e)] \left\{ U[(0, g_1)] U[(z_2, e)] \right\} U[(0, g_2)] \\
&\stackrel{\text{exchange two terms in curly bracket}}{=} \left\{ U[(z_1, e)] U[(\rho_{g_1}(z_2), e)] \right\} \left\{ U[(0, g_1)] U[(0, g_2)] \right\} \\
&\stackrel{\text{projective representations}}{=} \left\{ U[(z_1 + \rho_{g_1}(z_2), e)] \right\} \left\{ e^{i\nu(g_1, g_2)} U[(c(g_1, g_2), g_1 g_2)] \right\} \\
&\stackrel{\text{factorization}}{=} e^{i\nu(g_1, g_2)} U[(z_1 + \rho_{g_1}(z_2), e)] \left\{ U[(c(g_1, g_2), e)] U[(0, g_1 g_2)] \right\} \\
&\stackrel{\text{associativity}}{=} e^{i\nu(g_1, g_2)} \left\{ U[(z_1 + \rho_{g_1}(z_2), e)] U[(c(g_1, g_2), e)] \right\} U[(0, g_1 g_2)] \\
&= e^{i\nu(g_1, g_2)} \left\{ U[(z_1 + \rho_{g_1}(z_2) + c(g_1, g_2), e)] U[(0, g_1 g_2)] \right\} \\
&= e^{i\nu(g_1, g_2)} U[(z_1 + \rho_{g_1}(z_2) + c(g_1, g_2), g_1 g_2)] \\
&= e^{i\nu(g_1, g_2)} U[(z_1, g_1) \cdot (z_2, g_2)]. \tag{5.148a}
\end{aligned}$$

Since the right-hand sides of Eqs. (5.147) and (5.148a) must be equal, it follows that

$$e^{i\phi_1[(z_1, g_1), (z_2, g_2)]} = e^{i\nu(g_1, g_2)} \in \text{U}(1) \iff \phi_1[(z_1, g_1), (z_2, g_2)] = \nu(g_1, g_2) \pmod{2\pi}. \tag{5.149}$$

Chapter 6

Summary of results and outlook

The goal of this thesis was to generalize the boundary-based classification of one-dimensional fermionic IT and SPT phases with a generic on-site symmetry group G^{FP} from Ref. [31] to parafermions (PFs). The central result in Ref. [31] was that any one-dimensional fermionic IT or SPT phase with symmetry G^{FP} is completely characterized by a triplet of indices

$$([\nu, \rho], [\mu]) \tag{6.1}$$

that encode the transformation properties of Majorana boundary degrees of freedom. The $\mathbb{Z}_2$ -valued index $[\mu]$ records the parity of the number of Majorana modes localized on a boundary, distinguishing between intrinsically fermionic phases ($[\mu] = 1$) and phases admitting purely local fermionic Fock spaces on the boundary ($[\mu] = 0$). The remaining data $[(\nu, \rho)]$ classify projective representations of the symmetry group on the boundary and take values in suitable cohomology groups determined by G^{FP} .

The main result of this thesis is the construction of a classification in terms of a triplet of indices

$$[(\nu, \kappa), [\mu]) \tag{6.2}$$

that encode the transformation properties of parafermionic boundary degrees of freedom protected by the parafermionic symmetry group G^{PF} . Hereto, the $\mathbb{Z}_2$ -valued index $[\mu]$ records the parity of the number of parafermions on the boundary, distinguishing between intrinsically parafermionic phases ($[\mu] = 1$) and phases admitting purely local parafermionic Fock spaces on the boundary ($[\mu] = 0$). The remaining data $[(\nu, \kappa)]$ classify projective representations of the symmetry group on the boundary and take values in suitable cohomology groups determined by G^{PF} . In particular, the index ν classifies the projective representation of the symmetries $G^{\text{PF}}/\mathbb{Z}_N^{\text{PF}}$, while the index κ describes how these are married to the intrinsic parafermion charge symmetry.

We have defined, generalized, and rigorously characterized key structures of parafermionic systems, such as the local and global charge as well as their conjugation. The contents of this thesis thus establish a robust and complete foundation, from which we have begun to study the following open problems.

1. By design, IT and SPT phases are stable to “stacking”, by which is meant that the classifying topological indices accommodate a group composition law.¹ For example,

¹This is one rational for the use of the adjective invertible in “invertible topological phases”.

consider two decoupled Kitaev chains. Stacking them means that we take the tensor product of the domain of their Hamiltonians and we define a new Hamiltonian through the direct sum. Stacking n Kitaev chain in the symmetry class D produces a Hamiltonian that realizes the IT topological phase $n \bmod 2$. References [31] derived the stacking rules of the fermionic indices

$$([\nu, \rho]), [\mu]. \quad (6.3)$$

What are the stacking rules of the parafermionic indices

$$([\nu, \kappa]), [\mu]? \quad (6.4)$$

2. Unlike in Ref. [31], we assumed in this thesis that the symmetry group G^{PF} is represented by unitary operators. As complex conjugation implements the map

$$\omega_N \longrightarrow (\omega_N)^* = (\omega_N)^{N-1}, \quad (6.5)$$

imposing a symmetry in an anti-unitary way is related to understanding the effects of transformations under which

$$\omega_N \longrightarrow (\omega_N)^a \quad (6.6)$$

for some $a \in \mathbb{Z}_N$.

3. We defined our order- N parafermion algebras by fixing $\omega_N := \exp(2\pi i/N)$. Could we have chosen another primitive N -th root of identity? Can we construct $\mathbb{C}$ -algebra isomorphisms between them that are on-site?
4. Is there a $\mathbb{C}$ -algebra isomorphism such that

$$\mathcal{P}_N^m \cong \prod_{i=1}^k \mathcal{P}_{p_i^{r_i}}^m, \quad (6.7a)$$

where we make use on the right-hand side of the prime-power factorization

$$N = \oplus_{i=1}^k p_i^{r_i} \quad (6.7b)$$

Can we understand this from a mathematical point of view in terms of the Chinese Remainder Theorem? Is this related to the stacking of parafermionic phases?

5. How do we include symmetries that are not on-site, say translation or reflection symmetries?
6. Can we construct explicit parafermionic Hamiltonians that realize parafermionic IT and SPT phases.

Appendix A

Integer modulo N rings

We introduce some elementary results for arithmetics modulo N in the language of algebraic rings and related facts from number theory. A detailed treatment is be found in Lang [51].

General properties of integer modulo N rings.

Definition 20 (Groups). A *group* $(G, \cdot)$ is a set G equipped with a binary operation

$$\cdot : G \times G \rightarrow G \tag{A.1}$$

such that:

- associativity: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
- identity: $\exists e \in G$ with $e \cdot a = a \cdot e = a$,
- inverses: $\forall a \in G, \exists a^{-1} \in G$ with $a \cdot a^{-1} = a^{-1} \cdot a = e$.

Remark 33. A group G is said to be Abelian if the group operation is commutative, i.e., $a \cdot b = b \cdot a$ for all $a, b \in G$. In this case it is common to write ‘+’ for group product and ‘ $-a$ ’ for group inverse if $a \in G$.

Example 11 (The additive group $\mathbb{Z}_N$). For an integer $N \geq 1$, define

$$\mathbb{Z}_N = \{[0], [1], \dots, [N-1]\} \tag{A.2}$$

as the set of equivalence classes $[a]$ under $a \sim b \iff N \mid (a - b)$ (N divides $(a - b)$). With addition

$$[a] + [b] := [a + b], \tag{A.3}$$

$(\mathbb{Z}_N, +)$ is a finite Abelian group. It is isomorphic to the cyclic group of order N .

Definition 21 (Rings). A (*unital*) *ring* $(R, +, \cdot)$ is a set R with two binary operations such that:

- $(R, +)$ is an Abelian group,
- $(R, \cdot)$ is a monoid, meaning that:

- the multiplication ‘ $\cdot$ ’ is associative,
- there is an element 1 in R such that $a \cdot 1 = 1 \cdot a = a$ for all a in R (multiplicative identity)
- distributivity holds: $a(b + c) = ab + ac$, $(a + b)c = ac + bc$.

Remark 34. Note that there need not exist a multiplicative inverse for every element. A non-trivial ring where every non-zero element has a multiplicative inverse is called a *division ring*. If additionally the multiplication ‘ $\cdot$ ’ is commutative, then the ring is called a field.

Example 12 (The integer modulo N ring $\mathbb{Z}/N\mathbb{Z}$). Define multiplication on $\mathbb{Z}_N$ by

$$[a] \cdot [b] := [ab]. \quad (\text{A.4})$$

Then $\mathbb{Z}/N\mathbb{Z} := (\mathbb{Z}_N, +, \cdot)$ is a commutative ring with unit $[1]$. We will drop the equivalence class bracket ‘ $[\]$ ’ when it is clear from the context that we are considering integers modulo N .

Definition 22 (Units and zero-divisors). Let R be a ring.

- $u \in R$ is a *unit* if $\exists v \in R$ with $uv = vu = 1$.
- $a \in R \setminus \{0\}$ is a *zero-divisor* if $\exists b \neq 0$ with $ab = 0$.

The set of all units

$$R^\times := \{u \in R \mid \exists u^{-1} \in R\} \quad (\text{A.5})$$

forms a group under multiplication.

Proposition 11 (Units and zero-divisors in $\mathbb{Z}/N\mathbb{Z}$). Let $a \in \mathbb{Z}/N\mathbb{Z}$.

- a is a unit $\iff \gcd(a, N) = 1$ (a and N are coprime).
- a is a zero-divisor $\iff \gcd(a, N) \neq 1$ and $a \neq 0 \pmod N$.
- $\mathbb{Z}/N\mathbb{Z}$ is a field $\iff N$ is a prime number
- N is a prime number $\implies (\mathbb{Z}/N\mathbb{Z})^\times$ is isomorphic to the cyclic group of order $N - 1$.
- $|(\mathbb{Z}/N\mathbb{Z})^\times| = \varphi(N)$, where φ is Euler’s totient function counting the positive integers that are smaller and coprime to N .

Here, $\gcd(a, N)$ denotes the greatest common divisor of a and N .

Proof. Bézout’s identity [52,53] states that if a, b are integers with $\gcd(a, b) = d$, then there exists integers x, y such that

$$ax + by = d. \quad (\text{A.6})$$

Applied to the case where $a < b = N$ yields:

$$\gcd(a, N) = 1 \iff \exists x, y \in \mathbb{Z} : ax + Ny = 1 \quad (\text{A.7})$$

$$\implies ax + Ny \equiv ax = 1 \pmod{N}. \quad (\text{A.8})$$

If $\gcd(a, N) = d > 1$, then $a \cdot \frac{N}{d} \equiv 0 \pmod{N}$ with $\frac{N}{d} \neq 0$.

The remaining statements follow from by counting coprime numbers and using the fact that any group with a prime number of elements is isomorphic to the corresponding cyclic group. $\square$

Remark 35. The *Euclidean algorithm* allows one to efficiently compute $\gcd(a, N)$. The *related extended Euclidean algorithm* yields the multiplicative inverse a^{-1} of a unit $a \in \mathbb{Z}/N\mathbb{Z}$.

The multiplicative inverse of a unit in $\mathbb{Z}/N\mathbb{Z}$ may be easily computed using the Euler's totient function (assuming one has a table of values at hand). We cite one of Euler's many theorems.

Theorem 1 ((Fermat)-Euler Theorem). If a and N are coprime positive integers, then

$$a^{\varphi(N)} \equiv 1 \pmod{N}, \quad (\text{A.9})$$

where φ is Euler's totient function.

Corollary 3 (Multiplicative inverse via Euler's totient function). Let $a \in \mathbb{Z}/N\mathbb{Z}$ be a unit. Then

$$a^{-1} = a^{\varphi(N)-1} \in \mathbb{Z}/N\mathbb{Z}. \quad (\text{A.10})$$

Remark 36. The special case where N is prime is known as Fermat's little theorem. Pierre de Fermat first stated it in a letter from 1640, and provided the following compact proof [54]:

“Et cette proposition est généralement vraie en toutes progressions et en tous nombres premiers; de quoi je vous enverrois la démonstration, si je n'appréhendois d'être trop long.”

Euler provided the first published proof in 1736 [55] (although Leibniz had provided an unpublished version sometime before 1683). Almost thirty years later, Euler generalized the proof to the general non-prime case [56].

Structure of integer modulo N rings We state some structure theorems for finite Abelian groups and rings modulo integer N . These allow us to decompose arithmetic modulo N problems into simpler sub-problems.

Theorem 2 (Fundamental theorem of finite Abelian groups, Kronecker [1870]). 1. For two integers $n, m \geq 1$, the cyclic group decomposes into the direct sum

$$\mathbb{Z}_{nm} \cong \mathbb{Z}_n \times \mathbb{Z}_m \iff \gcd(n, m) = 1. \quad (\text{A.11})$$

2. Let G be a finite Abelian group of order $|G| = N$. Suppose N has the prime power decomposition $N = \prod_p p^{r_p}$, where the product runs over the distinct prime numbers p dividing N . Then G is isomorphic to a direct sum of the form

$$G \cong \prod_p \left(\mathbb{Z}_{p^{\lambda_{p,1}}} \times \cdots \times \mathbb{Z}_{p^{\lambda_{p,r_p}}} \right), \quad (\text{A.12})$$

where $\lambda_{p,i}$ are positive integers such that

$$\lambda_{p,1} \geq \lambda_{p,2} \geq \cdots \geq \lambda_{p,r_p} \geq 1, \quad \lambda_{p,1} + \lambda_{p,2} + \cdots + \lambda_{p,r_p} = r_p. \quad (\text{A.13})$$

The finite case was first proven by Leopold Kronecker in 1870 [57]. It was generalized to finitely generated Abelian groups by Emmy Noether in 1926 [58] (using only 14 lines).

Example 13 (Classification of some finite Abelian groups). Some examples for the classification of finite Abelian groups follow.

1. There exists up to isomorphisms exactly one Abelian group of order 15:

$$\mathbb{Z}_6 \cong \mathbb{Z}_3 \times \mathbb{Z}_5. \quad (\text{A.14})$$

2. There exists up to isomorphisms exactly two Abelian groups of order 4:

$$\mathbb{Z}_4, \quad \mathbb{Z}_2 \times \mathbb{Z}_2. \quad (\text{A.15})$$

3. For every prime number p , there exists up to isomorphisms exactly one Abelian group of order p :

$$\mathbb{Z}_p. \quad (\text{A.16})$$

4. For every prime power p^n , there exists up to isomorphisms exactly $p(n)$ Abelian group of order p^n , where $p(n)$ is the (number theoretical) partition function representing the number of possible partitions of a non-negative integer n .

We state a similar theorem relating to the decomposition of modulo integer N rings into direct sums [59,60].

Theorem 3 (Chinese Remainder Theorem, Qin Jiushao (秦九韶) [1247]). Let $n_1, \dots, n_k \geq 2$ be pairwise coprime ($\gcd(n_i, n_j) = 1$) integers greater than two, often called *moduli*. Define $N = \prod_{i=1}^k n_i$ to be the product of the moduli.

- Let $a_1, \dots, a_k \in \{0, 1, \dots, N-1\}$ be another set of integers defining the *system of congruences*

$$\begin{aligned} x &\equiv a_1 \pmod{n_1}, \\ &\vdots \\ x &\equiv a_k \pmod{n_k}, \end{aligned} \quad (\text{A.17})$$

where $x \geq 0$ is an unknown non-negative integer. Then there exists a solution x to the above system of congruences. Moreover, if x_1, x_2 are two possible solutions, then $x_1 \equiv x_2 \pmod{N}$.

- In term of algebraic rings, this is equivalent to the statement that the map

$$x \pmod{N} \mapsto (x \pmod{n_1}, \dots, x \pmod{n_k}) \quad (\text{A.18})$$

defines a ring isomorphism, yielding

$$\mathbb{Z}/N\mathbb{Z} \cong \prod_{i=1}^k \mathbb{Z}/n_i\mathbb{Z}. \quad (\text{A.19})$$

Corollary 4 (Structure of $(\mathbb{Z}/N\mathbb{Z})^\times$). Let

$$N = \prod_p p^{r_p} \quad (\text{A.20})$$

be the prime power decomposition. Then

$$\mathbb{Z}/N\mathbb{Z} \cong \prod_p \mathbb{Z}/p^{r_p}\mathbb{Z}. \quad (\text{A.21})$$

and

$$(\mathbb{Z}/N\mathbb{Z})^\times \cong \prod_p (\mathbb{Z}/p^{r_p}\mathbb{Z})^\times. \quad (\text{A.22})$$

We state the following theorem partly due Gauss which in combination with the above corollary lets us compute general unit groups of integer modulo N rings.

Proposition 12 (Prime power unit groups $(\mathbb{Z}/p^r\mathbb{Z})^\times$). We consider the unit group $(\mathbb{Z}/p^r\mathbb{Z})^\times$ of rings of integers modulo prime powers p^r ($r \geq 1$).

- For generic prime powers,

$$|(\mathbb{Z}/p^r\mathbb{Z})^\times| = \varphi(p^r) = p^{r-1}(p-1). \quad (\text{A.23})$$

- $(\mathbb{Z}/p^r\mathbb{Z})^\times$ is cyclic if p is an odd prime. More generally,

$$(\mathbb{Z}/n\mathbb{Z})^\times \cong \mathbb{Z}_{p^{r-1}(p-1)} \iff n = 1, 2, 4, p^r, 2p^r, \quad (\text{A.24})$$

where p is an odd prime and $r \geq 1$

- For powers of $p = 2$,

$$(\mathbb{Z}/2\mathbb{Z})^\times = \{1\}, \quad (\mathbb{Z}/4\mathbb{Z})^\times \cong \mathbb{Z}_2, \quad (\mathbb{Z}/2^r\mathbb{Z})^\times \cong \mathbb{Z}_2 \times \mathbb{Z}_{2^{r-2}} \quad (r \geq 3). \quad (\text{A.25})$$

Example 14 (Solutions to $a^2 \equiv 1 \pmod{N}$). We solve

$$a^2 \equiv 1 \pmod{N}. \quad (\text{A.26})$$

Using the Chinese remainder theorem, this reduces to solving

$$a^2 \equiv 1 \pmod{p^{r_p}} \quad (\text{A.27})$$

for each factor in the prime decomposition of $N = \prod_p p^{r_p}$.

Prime powers.

- For odd primes p :

$$a \equiv \pm 1 \pmod{p^{r_p}}. \quad (\text{A.28})$$

- For $p = 2$:

$$\begin{aligned} r_2 = 1 : & \quad a \equiv 1 \pmod{2}, \\ r_2 = 2 : & \quad a \equiv \pm 1 \pmod{4}, \\ r_2 \geq 3 : & \quad a \equiv \pm 1, \pm(1 + 2^{r_2-1}) \pmod{2^{r_2}}. \end{aligned} \quad (\text{A.29})$$

General solution. The full solution set is obtained by combining the solutions for each prime power via CRT. Hence, the number of solutions equals

$$\prod_p s_p, \quad (\text{A.30})$$

where s_p is the number of solutions modulo p^{r_p} .

In particular:

- each odd prime factor contributes 2 solutions,
- the 2-power contributes 1, 2, or 4 depending on r_2 .

Example 15 (Solutions to $a^2 \equiv 1 \pmod{12}$). Let $N = 12 = 2^2 \cdot 3$. We solve

$$a^2 \equiv 1 \pmod{12}.$$

We proceed via the Chinese remainder theorem.

Step 1: Reduction to prime powers. We solve the congruences

$$a^2 \equiv 1 \pmod{4}, \quad a^2 \equiv 1 \pmod{3}.$$

Step 2: Solutions modulo 4.

$$a^2 \equiv 1 \pmod{4} \iff a \equiv \pm 1 \pmod{4}.$$

Step 3: Solutions modulo 3.

$$a^2 \equiv 1 \pmod{3} \iff a \equiv \pm 1 \pmod{3}.$$

Step 4: Combine via Chinese remainder theorem. We combine all choices:

$$(a \pmod{4}, a \pmod{3}) \in \{\pm 1\} \times \{\pm 1\}.$$

Solving each system yields:

$$\begin{aligned} (1, 1) &\Rightarrow a \equiv 1 \pmod{12}, \\ (1, -1) &\Rightarrow a \equiv 5 \pmod{12}, \\ (-1, 1) &\Rightarrow a \equiv 7 \pmod{12}, \\ (-1, -1) &\Rightarrow a \equiv 11 \pmod{12}. \end{aligned}$$

Conclusion. The solutions are

$$a \equiv 1, 5, 7, 11 \pmod{12}.$$

Appendix B

Group cohomology

We introduce the algebraic notions necessary to define group cohomologies. Let M be a G -module via group action ρ in this section. We refer to [61] for a complete study of the cohomology of groups.

Definition 23 (Group actions and G -modules). Let $(G, \cdot)$ be an arbitrary group and let $(M, +)$ be an Abelian one. A *group action* of G on M is a group homomorphism

$$\rho_{(\cdot)} : G \longrightarrow \text{Aut}(M), \quad (\text{B.1})$$

where $\text{Aut}(M)$ is the group of automorphisms of M . We say that M is a G -module via the group action ρ .

Example 16 (Examples of G -modules). • Let G be a group and V be a vector space over some (algebraic) field K . A representation $U : G \rightarrow \text{GL}(V)$ makes V into G -module.

- Let $G = \mathbb{Z}_2$ and $M = \mathbb{Z}_N$ be the order-2 and order- N additive cyclic group. The map

$$\rho_g(z) := \begin{cases} z, & \text{if } g = 0, \\ -z & \text{if } g = 1, \end{cases} \quad (\text{B.2})$$

is a group action and makes $\mathbb{Z}_N$ into a $\mathbb{Z}_2$ -module.

Lemma 12 (Properties of group actions). Let M be a G_ρ -module via a group action ρ . Then

$$\rho_g(m_1 + m_2) = \rho_g(m_1) + \rho_g(m_2) \quad (\text{B.3a})$$

$$\rho_g(0) = 0 \quad \forall g \in G \quad (\text{B.3b})$$

$$\rho_{g_1}(\rho_{g_2}(m)) = \rho_{g_1 \cdot g_2}(m) \quad (\text{B.3c})$$

$$\rho_e(m) = m \quad \forall m \in M. \quad (\text{B.3d})$$

We enumerate all possible G -module structures for the integer modulo N group $\mathbb{Z}_N$. In particular, the group actions are nothing but multiplication by coprimes of N .

Lemma 13 (Group actions on $\mathbb{Z}_N$). Let $\rho_{(\cdot)}$ be a group action of G on $\mathbb{Z}_N$. Then, for all $g \in G$, there exists

$$a_g = 1, \dots, N-1, \quad \gcd(a_g, N) = 1, \quad (\text{B.4a})$$

such that for all $z \in \mathbb{Z}_N$

$$\rho_g(z) = z \cdot \rho_g(1) \equiv z \cdot a_g \in \mathbb{Z}_N. \quad (\text{B.4b})$$

In particular, for $g_1, g_2 \in G$,

$$a_{g_1} \cdot a_{g_2} \equiv a_{g_1 g_2} \pmod{N}. \quad (\text{B.4c})$$

Proof. By definition, $\rho_{(\cdot)}$ is a group homomorphism

$$\rho_{(\cdot)} : G \longrightarrow \text{Aut}(\mathbb{Z}_N). \quad (\text{B.5a})$$

Hence, for all $g \in G$,

$$\rho_g(z) = \rho_g\left(\sum_{i=1}^z 1\right) = \sum_{i=1}^z \rho_g(1) = z \cdot \rho_g(1), \quad (\text{B.5b})$$

using $\rho_g \in \text{Aut}(\mathbb{Z}_N)$. Moreover, automorphisms preserve the order of group elements, whence

$$\text{ord}(\rho_g(1)) = \text{ord}(1) = N. \quad (\text{B.5c})$$

Since $\text{ord}(a) = N$ if and only if $\gcd(a, N) = 1$, we infer that

$$\rho_g(z) = z \cdot \rho_g(1) = z \cdot a_g, \quad \text{for some } 0 < a_g < N, \quad \gcd(a_g, N) = 1. \quad (\text{B.5d})$$

The second statement follows from the fact that $\rho_{(\cdot)}$ is a group homomorphism, i.e., $\rho_{g_1} \circ \rho_{g_2} = \rho_{g_1 g_2}$ $\square$

Remark 37. It follows as a corollary that $\text{Aut}(\mathbb{Z}_N) \cong (\mathbb{Z}/N\mathbb{Z})^\times$, where the latter is the unit group of the integer modulo N ring $\mathbb{Z}/N\mathbb{Z}$ from **Appendix A**.

Definition 24 (n -cochain). For a G -module M , a n -cochain is a map

$$\begin{aligned} \phi : \underbrace{G \times \dots \times G}_{n \text{ times}} &\rightarrow M, \\ (g_1, g_2, \dots, g_n) &\mapsto \phi(g_1, g_2, \dots, g_n). \end{aligned} \quad (\text{B.6})$$

The set of all n -cochains from G^n to M is denoted by $C^n[G, M]$. By convention, $C^0[G, M] := M$. By endowing this set with pointwise addition as well as the appropriate inversion map for the cochains ϕ , we inherit from M the Abelian group structure $(C^n[G, M], +)$. We will use the shorthand $-\phi$ for the inverse of $\phi \in C^n[G, M]$.

Definition 25 (Coboundary Operator). Let the ρ be the group action of the G -module M . The *coboundary operator* δ_ρ^n is defined as the group homomorphism

$$\begin{aligned} \delta_\rho^n : C^n[G, M] &\rightarrow C^{n+1}[G, M], \\ \phi &\mapsto (\delta_\rho^n \phi), \end{aligned} \quad (\text{B.7})$$

that maps n -cochains to $(n + 1)$ -cochains such that

$$\begin{aligned} (\delta_\rho^n \phi)(g_1, \dots, g_{n+1}) &:= \rho_{g_1}(\phi(g_2, \dots, g_{n+1})) \\ &+ \sum_{i=1}^n (-1)^i \phi(g_1, \dots, \underline{g_i \cdot g_{i+1}}, \dots, g_{n+1}) \\ &- (-1)^n \phi(g_1, \dots, g_n). \end{aligned} \quad (\text{B.8})$$

Example 17 (Coboundary Operator for $n = 1$).

$$\begin{aligned} (\delta_\rho^1 \phi)(g_1, g_2) &= \rho_{g_1}(\phi(g_2)) + (-1)^1 \phi(g_1 \cdot g_2) - (-1)^1 \phi(g_1) \\ &= \rho_{g_1}(\phi(g_2)) - \phi(g_1 \cdot g_2) + \phi(g_1). \end{aligned} \quad (\text{B.9})$$

Example 18 (Coboundary Operator for $n = 2$).

$$\begin{aligned} (\delta_\rho^2 \phi)(g_1, g_2, g_3) &= \rho_{g_1}(\phi(g_2, g_3)) + (-1)^1 \phi(g_1 \cdot g_2, g_3) + (-1)^2 \phi(g_1, g_2 \cdot g_3) \\ &- (-1)^2 \phi(g_1, g_2) \\ &= \rho_{g_1}(\phi(g_2, g_3)) - \phi(g_1 \cdot g_2, g_3) + \phi(g_1, g_2 \cdot g_3) - \phi(g_1, g_2). \end{aligned} \quad (\text{B.10})$$

Definition 26 (n -cocycles, n -coboundaries). For $G, M, \rho, \delta_\rho^n$ as above, we define the set of n -cocycles

$$Z^n[G, M_\rho] := \ker(\delta_\rho^n) = \{ \phi \in C^n(G, M) \mid \delta_\rho^n \phi = 0 \}, \quad (\text{B.11})$$

and the set of n -coboundaries

$$B^n[G, M_\rho] := \text{im}(\delta_\rho^{n-1}) = \{ \phi \in C^n(G, M) \mid \phi = \delta_\rho^{n-1} \phi', \phi' \in C^{n-1}(G, M) \}, \quad (\text{B.12})$$

with the convention

$$B^0[G, M_\rho] := 0. \quad (\text{B.13})$$

We noted the ρ -dependency of the action of the coboundary operator δ_ρ^n on M above by writing M_ρ . From the first group isomorphism theorem as well as the fact that $C^n(G, M)$ is Abelian, we conclude that both the n -cocycles and n -coboundaries form normal subgroups of the n -cochains.

The role of the coboundaries becomes clear when considering

Proposition 13 (Coboundaries as Normal Subgroups of Cocycles). For $n \in \mathbb{N}$ let δ_ρ^n , $Z^n[G, M_\rho]$ and $B^n[G, M_\rho]$ be the coboundary operator, n -cocycle and n -coboundary group respectively. Then

$$\phi = \delta_\rho^{n-1} \phi' \implies \delta_\rho^n \phi = 0, \quad (\text{B.14})$$

holds. This is equivalent to $\delta_\rho^n \circ \delta_\rho^{n-1} \equiv 0$. Moreover, the n -coboundaries are a normal subgroup of the n -cocycles,

$$B^n[G, M_\rho] \trianglelefteq Z^n[G, M_\rho]. \quad (\text{B.15})$$

To conclude, we define the

Definition 27 (*n*-th Cohomology Group). The *n*-th cohomology group is defined as the quotient of the *n*-cocycles by the *n*-coboundaries,

$$H^n[G, M_\rho] := Z^n[G, M_\rho] / B^n[G, M_\rho]. \quad (\text{B.16})$$

By **Proposition 13**, this quotient is well-defined.

Remark 38 (Terminology). Constraints of the form $\delta_\rho^n \phi = 0$ are called *n*-cocycle conditions. We shall encounter in this work cocycle conditions deviating from the *regular* ones defined in this appendix, e.g., by Eqs. (B.9) and (B.10). When the individual addends match up to multiplicative factors (e.g., Eq. (5.39d)), we say that we have a *twisted* cocycle condition. If there is an additional non-cochain term (e.g., Eq. (5.52c)), we say that the cocycle condition is *modified*.

To end, we consider the following remark regarding a more general definition of the cohomology group from algebraic topology.

Remark 39 (Cochains and Cohomology in Algebraic Topology). We sketch an alternative definition that is more commonly used in algebraic topology, cf., [62]. First, we define a *cochain complex* $(A_\bullet, d_\bullet)$ as a sequence of Abelian groups $\cdots A_0, A_1, A_2, A_3, \cdots$ connected by homomorphisms, so-called coboundary operators, $d_n : A_{n-1} \rightarrow A_n$ such that $d_n \circ d_{n-1} = 0$. We may represent a cochain complex as

$$\cdots \xrightarrow{d_0} A_0 \xrightarrow{d_1} A_1 \xrightarrow{d_2} A_2 \xrightarrow{d_3} A_3 \xrightarrow{d_4} \cdots \quad (\text{B.17})$$

As one can see, for a group G and Abelian group M , the *n*-cochains $C^n[G, M]$ defined in **Definition 24** fulfill the requirements of the Abelian groups A_n . The same holds for the coboundary operator δ_ρ^n from **Definition 25**. Finally, the cohomology group is defined as $H^n := \ker(d^n) / \text{im}(d^{n-1})$, which coincides with our definition through the cocycle group $Z^n[G, M_\rho]$ and coboundary group $B^n[G, M_\rho]$.

Appendix C

Group extensions

We review the definition of group extensions and present a classification result. Our goal is to give a general factorization of a non-simple group G into a normal Abelian subgroup K and quotient G/K . We follow the presentation of Ref. [63]. We refer to Ref. [64] for a more pedagogical treatment of the underlying mathematics and the relevant appendix of Ref. [65] for a compact review tailored to symmetry topological field theory, i.e., an invertible topological field theory equipped with a global symmetry and a consistent coupling to background symmetry gauge fields.

Given a group G with normal subgroup K , the quotient G/K captures part of the structure of G , and one may view G as being built from K and G/K in a way that is generally more subtle than a direct product. The study of group extensions asks the inverse question: given a group K and a group Q , what can be said about the possible groups G containing K as a normal subgroup with quotient $G/K \cong Q$. This is known as the extension problem, and was first stated by Otto Hölder [66] in 1889:¹

Zu der Aufgabe, eine Gruppe in zwei Factoren zu spalten, kann man sich die umgekehrte stellen: Gegeben zwei Gruppen als Factoren, aus denselben eine Gruppe als Product zusammenzusetzen. Diese Aufgabe lässt manchmal mehrere Lösungen zu, ich hoffe die selbe bei einer anderen Gelegenheit zu behandeln.

Definition 28 (Group extensions). If K and Q are groups, then a (group) *extension* of Q by K is a group G with a normal subgroup K_1 such that

$$K_1 \cong K, \tag{C.1a}$$

$$G/K_1 \cong Q. \tag{C.1b}$$

We say that G is an *extension of Q by K* .

An extension of Q by K is said to be *split* if there exists a group homomorphism $s : Q \rightarrow G$ such that the composition

$$Q \xrightarrow{s} G \xrightarrow{\pi} G/K_1 \tag{C.2}$$

is the identity map on Q , where $\pi : G \rightarrow G/K_1$ denotes the canonical projection. Otherwise, the extension is said to be *non-split*.

¹See Ref. [67] for a historical overview.

An extension of Q by K is said to be a *central extension* if the normal subgroup K_1 lies in the center of G .

Remark 40. It is common to find the reverse terminology *extensions of K by Q* .

Example 19 (Direct product as trivial group extension). Any direct product $K \times Q$ is a trivial extension of Q by K (in both terminologies). This serves as the definition of a trivial extension: An extension of Q by K is trivial if it is the direct product $K \times Q$.

Example 20 (Semi-direct product as split group extension). Any semi-direct product $K \rtimes Q$ is a split extension of Q by K .

Example 21 (Discrete Heisenberg group as non-split central extension). Let

$$K = \mathbb{Z} \cong \left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mid c \in \mathbb{Z} \right\} \quad (\text{C.3})$$

and

$$Q = \mathbb{Z}^2 \cong \left\{ \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{Z} \right\} \quad (\text{C.4})$$

The discrete Heisenberg group (sometimes called the integral Heisenberg group)

$$G = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \mid a, b, c \in \mathbb{Z} \right\} \quad (\text{C.5})$$

is a central extension of $\mathbb{Z}^2$ by $\mathbb{Z}$ that is non-split. It is a non-Abelian group.

Example 22 (The group $\mathbb{Z}_8$). The group $\mathbb{Z}_8$ is also a central extension of $\mathbb{Z}_4$ by $\mathbb{Z}_2$ that is non-split, thereby demonstrating that non-trivial extensions need not be non-Abelian.

Group extensions may be equivalently described in terms of short exact sequences.

Definition 29 (Short exact sequences). Let

$$\cdots \rightarrow A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \rightarrow \cdots \quad (\text{C.6a})$$

be a sequence of groups and homomorphisms. The sequence is called *exact at A_i* if

$$\text{im}(f_{i-1}) = \ker(f_i). \quad (\text{C.6b})$$

It is called a *short exact sequence* if it is exact at every term A_i .

A sequence of groups and homomorphisms

$$1 \rightarrow N \xrightarrow{\iota} E \xrightarrow{\pi} G \rightarrow 1, \quad (\text{C.6c})$$

where ‘1’ denotes the trivial group $\{e\}$, is called *short exact* if it is exact. It follows from the isomorphism theorems for groups that

$$\iota \text{ is injective, } \pi \text{ is surjective, } \ker(\pi) = \text{im}(\iota). \quad (\text{C.6d})$$

Proposition 14 (Equivalent definition of group extensions via short exact sequences). An extension of a group Q by a group K is a short exact sequence of groups

$$1 \rightarrow K \xrightarrow{\iota} G \xrightarrow{\pi} Q \rightarrow 1, \quad (\text{C.7})$$

where ι is injective, π is surjective, and $\ker(\pi) = \text{im}(\iota) \cong K$.

This allows us to define a notion of equivalent extensions.

Definition 30 (Equivalent group extensions). Let

$$1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1 \quad (\text{C.8a})$$

be an extension of Q by K . A second extension

$$1 \rightarrow K \rightarrow G' \rightarrow Q \rightarrow 1 \quad (\text{C.8b})$$

is said to be *equivalent* to the above if there exists a group homomorphism $G \rightarrow G'$ such that the diagram

$$\begin{array}{ccccccccc} 1 & \rightarrow & K & \rightarrow & G & \rightarrow & Q & \rightarrow & 1 \\ & & \parallel & & \downarrow & & \parallel & & \\ 1 & \rightarrow & K & \rightarrow & G' & \rightarrow & Q & \rightarrow & 1 \end{array} \quad (\text{C.8c})$$

commutes. Such a map is necessarily an isomorphism.

The extension problem asks, given two groups K and Q , to classify all group structures G that fit into a short exact sequence

$$1 \rightarrow K \rightarrow G \rightarrow Q \rightarrow 1,$$

up to the notion of equivalence introduced in **Definition 30**. One should therefore ask what is meant by “determining” the structure of such groups.

For our purposes, this amounts to classifying all possible group laws on G that realize K as a normal subgroup with quotient isomorphic to Q , modulo group isomorphisms. **Definition 30** thus naturally leads to the problem of classifying equivalence classes of group extensions.

This problem was solved by Otto Schreier in 1926 [68], who showed that group extensions of Q by K up to equivalence are classified by suitable ‘factor systems’. In modern language, these equivalence classes are described by the second cohomology groups $H^2[Q, K_\rho]$.

We restrict ourselves to extensions of Q by Abelian groups K . We proceed by constructing from Q and K a well-defined class of group extensions. We then conclude that our construction exhausts all possible group extensions (up to isomorphism) by citing Schreier’s theorem.

Definition 31 (Group extension associated to a cocycle (ρ, c)). Fix the additively written Abelian group K and general group Q . Let the group homomorphism

$$\rho : Q \rightarrow \text{Aut}(K) \quad (\text{C.9a})$$

define an action of Q on K , and let

$$c : Q \times Q \rightarrow K \quad (\text{C.9b})$$

be a normalized 2-cocycle, i.e.,

$$\rho_{q_1}(c(q_2, q_3)) - c(q_1 q_2, q_3) + c(q_1, q_2 q_3) - c(q_1, q_2) = 0 \quad (\text{C.9c})$$

for all $q_1, q_2, q_3 \in Q$ with the normalization conditions

$$c(q, e) = c(e, q) = 0 \quad \left(\implies c(e, e) = 0 \right) \quad (\text{C.9d})$$

for all $q \in Q$. Define on $K \times Q$ the binary operation

$$(k_1, q_1) \cdot (k_2, q_2) := (k_1 + \rho_{q_1}(k_2) + c(q_1, q_2), q_1 q_2). \quad (\text{C.9e})$$

The pair (ρ, c) is known as a *factor system*.

Proposition 15 (Properties of group extensions associated to a cocycle (ρ, c)). Let

$$G_{(\rho, c)} := (K \times Q, \cdot)$$

with the binary product ‘ $\cdot$ ’ given in **Definition 31**. The following statements hold.

1. $G_{(\rho, c)}$ is a group.
2. $K \times \{e\}$ is a normal subgroup of $G_{(\rho, c)}$.
3. $K \cong K \times \{e\}$ and $Q \cong G_{(\rho, c)} / (K \times \{e\})$.

Hence, $G_{(\rho, c)}$ is a group extension of Q by K .

Proof. 1. We begin with the proof that $G_{(\rho, c)}$ is a group.

Associativity Let $(k_1, q_1), (k_2, q_2), (k_3, q_3) \in K \times Q$. On the one hand,

$$\begin{aligned} & ((k_1, q_1) \cdot (k_2, q_2)) \cdot (k_3, q_3) \\ &= (k_1 + \rho_{q_1}(k_2) + c(q_1, q_2), q_1 q_2) \cdot (k_3, q_3) \\ &= (k_1 + \rho_{q_1}(k_2) + c(q_1, q_2) + \rho_{q_1 q_2}(k_3) + c(q_1 q_2, q_3), q_1 q_2 q_3). \end{aligned} \quad (\text{C.10a})$$

On the other hand,

$$\begin{aligned} & (k_1, q_1) \cdot ((k_2, q_2) \cdot (k_3, q_3)) \\ &= (k_1, q_1) \cdot (k_2 + \rho_{q_2}(k_3) + c(q_2, q_3), q_2 q_3) \\ &= (k_1 + \rho_{q_1}(k_2 + \rho_{q_2}(k_3) + c(q_2, q_3)) + c(q_1, q_2 q_3), q_1 q_2 q_3) \\ &= (k_1 + \rho_{q_1}(k_2) + \rho_{q_1 q_2}(k_3) + \rho_{q_1}(c(q_2, q_3)) + c(q_1, q_2 q_3), q_1 q_2 q_3), \end{aligned} \quad (\text{C.10b})$$

where we used that ρ is a group action, i.e.,

$$\rho_{q_1} \circ \rho_{q_2} = \rho_{q_1 q_2}. \quad (\text{C.10c})$$

Thus, associativity reduces to the equality

$$c(q_1, q_2) + c(q_1 q_2, q_3) = \rho_{q_1}(c(q_2, q_3)) + c(q_1, q_2 q_3), \quad (\text{C.10d})$$

which is exactly the 2-cocycle condition. Hence $\cdot$ is associative.

Identity We claim that $(0, e)$ is the identity element, where 0 denotes the neutral element of K and e that of Q . Indeed, for every $(k, q) \in K \times Q$,

$$(0, e) \cdot (k, q) = (0 + \rho_e(k) + c(e, q), eq) = (k, q), \quad (\text{C.11a})$$

since $\rho_e = \text{id}_K$ and $c(e, q) = 0$. Likewise,

$$(k, q) \cdot (0, e) = (k + \rho_q(0) + c(q, e), qe) = (k, q), \quad (\text{C.11b})$$

since $\rho_q(0) = 0$ and $c(q, e) = 0$. Therefore $(0, e)$ is the identity.

Inverse Let $(k, q) \in K \times Q$. We claim that

$$(k, q)^{-1} = (-\rho_{q^{-1}}(k + c(q, q^{-1})), q^{-1}). \quad (\text{C.12a})$$

First, we compute the right product,

$$\begin{aligned} (k, q) \cdot (-\rho_{q^{-1}}(k + c(q, q^{-1})), q^{-1}) &= \left(k + \rho_q(-\rho_{q^{-1}}(k + c(q, q^{-1}))) + c(q, q^{-1}), e \right) \\ &= \left(k - (k + c(q, q^{-1})) + c(q, q^{-1}), e \right) \\ &= (0, e). \end{aligned} \quad (\text{C.12b})$$

To verify the left product, we first derive the identity

$$c(q^{-1}, q) = \rho_{q^{-1}}(c(q, q^{-1})). \quad (\text{C.12c})$$

Indeed, applying the cocycle condition to (q^{-1}, q, q^{-1}) yields

$$\rho_{q^{-1}}(c(q, q^{-1})) - c(e, q^{-1}) + c(q^{-1}, e) - c(q^{-1}, q) = 0, \quad (\text{C.12d})$$

and normalization gives $c(e, q^{-1}) = c(q^{-1}, e) = 0$, proving the claim. We are ready to compute the left product,

$$\begin{aligned} (-\rho_{q^{-1}}(k + c(q, q^{-1})), q^{-1}) \cdot (k, q) &= \left(-\rho_{q^{-1}}(k + c(q, q^{-1})) + \rho_{q^{-1}}(k) + c(q^{-1}, q), e \right) \\ &= \left(-\rho_{q^{-1}}(c(q, q^{-1})) + c(q^{-1}, q), e \right) \\ &= (0, e). \end{aligned} \quad (\text{C.12e})$$

Hence the displayed element is both a left and a right inverse of (k, q) .

We have shown that $\cdot$ is associative, that $(0, e)$ is an identity, and that every element admits an inverse. Therefore $G_{(\rho, c)} = (K \times Q, \cdot)$ is a group.

2. We move to normality of $K \times \{e\}$. We first show that $K \times \{e\}$ is a subgroup.

Closure Let $(k_1, e), (k_2, e) \in K \times \{e\}$. Then

$$(k_1, e) \cdot (k_2, e) = (k_1 + \rho_e(k_2) + c(e, e), ee) = (k_1 + k_2, e), \quad (\text{C.13})$$

since $\rho_e = \text{id}_K$ and $c(e, e) = 0$. Hence the product again lies in $K \times \{e\}$.

Identity The identity element of $G_{(\rho, c)}$ is $(0, e)$, and clearly

$$(0, e) \in K \times \{e\}. \quad (\text{C.14})$$

Inverses Let $(k, e) \in K \times \{e\}$. By the inverse formula in $G_{(\rho, c)}$,

$$(k, e)^{-1} = (-\rho_{e^{-1}}(k + c(e, e^{-1})), e^{-1}). \quad (\text{C.15})$$

Since $e^{-1} = e$, $\rho_e = \text{id}_K$, and $c(e, e) = 0$, this becomes

$$(k, e)^{-1} = (-k, e) \in K \times \{e\}. \quad (\text{C.16})$$

Thus $K \times \{e\}$ is a subgroup.

Normality Let $(k, q) \in G_{(\rho, c)}$ and $(k', e) \in K \times \{e\}$. We show that

$$(k, q) \cdot (k', e) \cdot (k, q)^{-1} \in K \times \{e\}. \quad (\text{C.17})$$

First,

$$\begin{aligned} (k, q) \cdot (k', e) &= (k + \rho_q(k') + c(q, e), qe) \\ &= (k + \rho_q(k'), q), \end{aligned} \quad (\text{C.18})$$

since $c(q, e) = 0$. Next, using

$$(k, q)^{-1} = (-\rho_{q^{-1}}(k + c(q, q^{-1})), q^{-1}), \quad (\text{C.19})$$

we obtain

$$\begin{aligned} &(k + \rho_q(k'), q) \cdot (-\rho_{q^{-1}}(k + c(q, q^{-1})), q^{-1}) \\ &= (k + \rho_q(k') + \rho_q(-\rho_{q^{-1}}(k + c(q, q^{-1}))) + c(q, q^{-1}), e) \\ &= (k + \rho_q(k') - (k + c(q, q^{-1})) + c(q, q^{-1}), e) \\ &= (\rho_q(k'), e). \end{aligned} \quad (\text{C.20})$$

Since $\rho_q \in \text{Aut}(K)$, we have $\rho_q(k') \in K$, and therefore

$$(k, q) \cdot (k', e) \cdot (k, q)^{-1} = (\rho_q(k'), e) \in K \times \{e\}. \quad (\text{C.21})$$

Hence $K \times \{e\}$ is normal in $G_{(\rho, c)}$.

3. We prove the two statements separately.

Step 1: The groups K and $K \times \{e\}$ are isomorphic Define

$$\iota : K \rightarrow K \times \{e\}, \quad k \mapsto (k, e). \quad (\text{C.22})$$

This map is clearly bijective. It remains to check that it is a group homomorphism. For $k_1, k_2 \in K$,

$$\iota(k_1 + k_2) = (k_1 + k_2, e). \quad (\text{C.23})$$

On the other hand,

$$\begin{aligned} \iota(k_1) \cdot \iota(k_2) &= (k_1, e) \cdot (k_2, e) \\ &= (k_1 + \rho_e(k_2) + c(e, e), ee) \\ &= (k_1 + k_2, e), \end{aligned} \quad (\text{C.24})$$

since $\rho_e = \text{id}_K$ and $c(e, e) = 0$. Hence

$$\iota(k_1 + k_2) = \iota(k_1) \cdot \iota(k_2), \quad (\text{C.25})$$

so ι is an isomorphism. Therefore

$$K \cong K \times \{e\}. \quad (\text{C.26})$$

Step 2: The quotient $G_{(\rho, c)}/(K \times \{e\})$ is isomorphic to Q Define

$$\pi : G_{(\rho, c)} \rightarrow Q, \quad (k, q) \mapsto q. \quad (\text{C.27})$$

We first show that π is a group homomorphism. Let $(k_1, q_1), (k_2, q_2) \in G_{(\rho, c)}$. Then

$$\begin{aligned} \pi((k_1, q_1) \cdot (k_2, q_2)) &= \pi(k_1 + \rho_{q_1}(k_2) + c(q_1, q_2), q_1 q_2) \\ &= q_1 q_2 \\ &= \pi(k_1, q_1) \pi(k_2, q_2). \end{aligned} \quad (\text{C.28})$$

Thus π is a homomorphism. It is surjective, since for every $q \in Q$ one has

$$\pi(0, q) = q. \quad (\text{C.29})$$

Its kernel is given by

$$\ker(\pi) = \{(k, q) \in G_{(\rho, c)} \mid q = e\} = K \times \{e\}. \quad (\text{C.30})$$

Therefore, by the first isomorphism theorem,

$$G_{(\rho, c)}/(K \times \{e\}) \cong Q. \quad (\text{C.31})$$

This proves the claim. □

As we saw in the proof, the associativity of groups constrained c to be a 2-cocycle in $Z^2[Q, K_\rho]$. In view of the group cohomology discussed in **Appendix B**, one is tempted to ask what happens to c if it is shifted by a coboundary, i.e.,

$$c' := c + \delta^1 b, \quad b \in C^1[Q, K_\rho]. \quad (\text{C.32})$$

It turns out that the resulting groups are isomorphic,

$$G_{(c', \rho)} \cong G_{(c, \rho)}. \quad (\text{C.33})$$

In essence, changing c by a coboundary amounts to changing the representatives of Q in $G_{(\rho, c)}$. Note that this shows that choosing c to be a normalized 2-cocycle does not lead to a loss of generality.

We omit the proof of the above fact, and state the full classification result for extensions by Abelian groups.

Theorem 4 (Classification of group extensions by Abelian groups, Schreier [1926]). Let K be an Abelian group and Q a general group. Fix a group action ρ of Q on K . The set of group extensions of Q by K giving rise to the group action ρ are in one-to-one correspondence with the second group cohomology

$$H^2[Q, K_\rho]. \quad (\text{C.34})$$

Corollary 5. Given is an Abelian group K and a general group Q . All group extensions of Q by K are isomorphic to

$$G_{(\rho, c)} = (K \times Q, \bullet) \quad (\text{C.35})$$

for some group action $\rho : Q \rightarrow \text{Aut}(K)$ and 2-cocycle $c \in Z^2[Q, K_\rho]$.

Remark 41. We conclude with some nomenclature. Let ρ be a group of Q on K and let $[c] \in H^2[Q, K_\rho]$ denote the equivalence class of a 2-cocycle c . We say that the group extension of Q by K isomorphic to the group $G_{(\rho, c)}$ is

- *trivial*, if $[c] = 0$ and $\rho_{(\cdot)} = \text{Id}$. This corresponds to direct products $K \times Q$.
- *central*, if $[c] \neq 0$ and $\rho_{(\cdot)} = \text{Id}$. This corresponds to groups where K is a subgroup of the extension's center (elements of K commute with all other group elements).
- *split*, if $[c] = 0$ and $\rho_{(\cdot)} \neq \text{Id}$. This correspond to groups that can be written as semi-direct products $K \rtimes Q$.

Appendix D

Projective representations

We introduce the theory of linear and projective representations of groups. The classification of projective representations is stated in the group cohomology framework of **Appendix B**. We then explain why projective representations are warranted in the context of quantum mechanics. In particular, we show how the results of **Appendix C** can be used to realize anti-unitary projective representations. We refer to [63] for an in-depth study of projective representations in the context of group cohomology and extensions.

In the main body of this thesis, we will introduce projective representations which allow non-scalar central factors.

Projective Representations of Groups

We begin by defining linear and projective representations of a group G .

Definition 32 (Linear representation). A *linear representation* of a group G onto a vector space V is a group homomorphism

$$\rho : G \longrightarrow \mathrm{GL}(V), \quad (\text{D.1})$$

i.e.,

$$\rho(g) \rho(h) = \rho(gh), \quad (\text{D.2})$$

holds. Here, $\mathrm{GL}(V)$ is the set of invertible matrices over V .

We recall that the center of a group is the collection of elements that commute with all other elements of the group. This set is a normal subgroup. For a vector space V over a field F , the center of $\mathrm{GL}(V)$ is the set $F^* = \{rI \mid r \in F \setminus \{0\}\}$, where I is the identity on V .

Definition 33 (Projective representation). A *projective representation* of a group G on a vector space V is a group homomorphism

$$\tilde{\rho} : G \longrightarrow \mathrm{GL}(V)/F^*, \quad (\text{D.3})$$

where F^* is the center of $\mathrm{GL}(V)$. In more concrete terms, this is a map

$$\tilde{\rho}' : G \longrightarrow \mathrm{GL}(V), \quad (\text{D.4})$$

satisfying the group homomorphism property up to a multiplicative constant, i.e.,

$$\tilde{\rho}'(g)\tilde{\rho}'(h) = c(g, h)\tilde{\rho}'(gh), \quad \forall g, h \in G, \quad (\text{D.5})$$

where c is a map from $G \times G$ to $K^\times := K \setminus \{0\}$. Here, K is the underlying field of the vector space V and $0 \in K$ is its additive identity.

Remark 42. We shall call a map c satisfying Eq. (D.5) the *Schur-multiplier* of the projective representation ρ . As will be shown, Schur-multipliers can be identified with elements of the second group cohomology. In the mathematical literature, the term ‘Schur-multiplier’ refers not to a single 2-cocycle c , but to its entire equivalence class $[c]$.

If $c = 1$, then the projective representation is a linear one. Moreover, since $K^\times$ is a group, we see that c is in fact a 2-cochain when taking the trivial group action, see **Definition 24**. In fact, due to the associativity of $GL(V)$, we have the following lemma.

Lemma 14 (Projective representations and 2-cocycles). The Schur-multiplier associated to a projective representation is a 2-cocycle.

Proof. Let c be the 2-cochain for a trivial group action associated to a projective representation ρ . Since $GL(V)$ is associative, for any $g, h, k \in G$,

$$\begin{aligned} 0 &\stackrel{!}{=} \left(\rho(g)\rho(h) \right) \rho(k) - \rho(g) \left(\rho(h)\rho(k) \right) \\ &= - \left(c(h, k) c(g, h k) - c(g, h) c(g h, k) \right) \rho(g h k), \end{aligned} \quad (\text{D.6})$$

whence

$$c(h, k) c(g, h k) - c(g, h) c(g h, k) = 0 \quad (\text{D.7})$$

must hold. By **Definition 26** and **Example B.10**, we conclude that c is a 2-cocycle. $\square$

Remark 43. Can a projective representation be recovered from a linear one and vice versa? If we begin with a linear representation $\rho : G \rightarrow GL(V)$, we can find a projective representation $\tilde{\rho}$ of G by composing

$$\tilde{\rho} : G \xrightarrow{\rho} GL(V) \xrightarrow{\iota} GL(V)/F^*, \quad (\text{D.8})$$

where F^* is again the center of $GL(V)$ and $\iota : GL(V) \rightarrow GL(V)/F^*$ is the canonical projection into the quotient. The reverse process is called *lifting* a projective representation of a group to a linear one. In general, this is not possible. However, sufficient conditions to do so can be found using group cohomology.

Let us define projectively equivalent representations.

Definition 34 (Equivalent projective representation). Let V_1, V_2 be two vector spaces over a shared field K . Two projective representations ρ_1, ρ_2 of a group G onto V_1 and V_2 respectively are called *equivalent*, if there exists a vector field isomorphism $F : V_1 \rightarrow V_2$ and a function $\theta : G \rightarrow K^\times$ such that

$$F(\rho_1(g) \cdot v_1) = \theta(g)(\rho_2 \cdot F(v_1)) \quad (\text{D.9})$$

This is an equivalence relation.

In essence, two projective representations are equivalent if they are the same modulo a multiplicative factor as well as a change-of-basis transformation. As the name indicates, this is an equivalence relation on the set of projective representations of a group G on isomorphic vector spaces.

We state a theorem that ties projective representations with the second cohomological group.

Theorem 5 (Classification of projective representations). Let ρ_1, ρ_2 be two projective representations associated to the 2-cocycles c_1, c_2 respectively. Let $H^2[G, K^\times]$ be the second cohomology group of trivial group action. Then, if ρ_1 and ρ_2 are projectively equivalent, then

$$[\rho_1] = [\rho_2] \in H^2[G, K^\times]. \quad (\text{D.10})$$

Hence, each class of projectively equivalent can be associated to a well-defined element of the second cohomology group. In other words, they can be used to classify them.

This can be proven by computing the associated 2-cocycle of an equivalent representation by substituting in Eq. (D.5). Having done this, we observe that the emerging 2-cocycles differ by a coboundary. In fact, we obtain a criterion to show that two projective representations are not equivalent. By contraposition,

Corollary 6. Let ρ_1, ρ_2 be two projective representations associated to the 2-cocycles c_1, c_2 respectively. Let $H^2[G, K^\times]$ be the second cohomology group of trivial group action. Then

$$[\rho_1] \neq [\rho_2] \implies \rho_1 \text{ and } \rho_2 \text{ are not projectively equivalent.} \quad (\text{D.11})$$

Remark 44. As seen in **Appendix C**, the second cohomology group $H^2[G, K^\times]$ can be used to classify the central group extensions of G by the central factors $K^\times$. Hence, we may identify projective representations of a group G onto a vector space over the field K with central group extensions.

From **Theorem 5**, we see that if a projective representation is equivalent to a linear one, then the associated 2-cocycle must be a coboundary. In fact, it can be shown that the reverse holds, i.e.,

Proposition 16. For a projective representation there exists a linear one if and only if the associated 2-cocycle is a coboundary.

Note that the proposition only states that *there exists an* equivalent linear representation. This can be seen from the fact that not all linear representations are projectively equivalent. For example,

$$\rho_1, \rho_2 : \mathbb{Z} \rightarrow \text{GL}(\mathbb{C}^2), \quad \rho_1(g) = \begin{bmatrix} +1 & 0 & 0 \\ 1 & +1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \rho_2(g) = \begin{bmatrix} +1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad (\text{D.12})$$

are both linear, but not projectively equivalent. It also shows, that the reverse of **Theorem 5** does not hold. Thus, 2-cocycles can a priori not be used to show that projective representations are equivalent.

Projective Representation in Quantum Mechanics

We now consider a collection of requirements for projective representations inspired by quantum mechanics. Let G be a group that encodes the symmetries of a model and $\mathbb{C}^n$ be the vector space of our representation. Let $e \in G$ and $\mathbb{1} \in \text{GL}(\mathbb{C}^n)$ both be the respective identity elements. A projective representation $U : G \rightarrow \text{GL}(\mathbb{C}^n)$ must obey

$$U(e) = \mathbb{1} \tag{D.13a}$$

$$U(g) U(h) = e^{i\phi(g,h)} U(gh), \quad \phi : G \times G \rightarrow \mathbb{R}/2\pi\mathbb{Z}, \tag{D.13b}$$

$$\left(U(g) U(h) \right) U(f) = U(g) \left(U(h) U(f) \right), \tag{D.13c}$$

for all $g, h, f \in G$. These conditions are motivated by our desire that representation be normalized regarding the identity elements, as well as preserving associativity, and the fact that in quantum mechanics, states are defined up to a multiplicative phase. Since a projective representation can be thought of as a map to the projective linear group of $\text{GL}(\mathbb{C}^n)$, these conditions are not an important constraint. Due to the normalization,

$$\phi(e, g) = \phi(g, e) = 0 \quad \forall g \in G, \tag{D.13d}$$

holds.

Note that by identifying the group $(\mathbb{R}/2\pi\mathbb{Z}, +)$ with the unitary group $U(1)$, ϕ can be interpreted as a 2-cochain in $C^2[G, U(1)]$. By [Lemma 14](#), these are in fact 2-cocycles. Furthermore, let $\mathbf{c} : G \rightarrow (\{+1, -1\}, \cdot) \cong \mathbb{Z}_2$ be the group homomorphism inducing the group action required to define the coboundary operator $\delta_{\mathbf{c}}^1$, see [Example B.9](#). Here, the symbol $\mathbf{c} = \pm 1$ dictates whether the representation is unitary or anti-unitary, i.e., $\mathbf{c} = +1$ or $\mathbf{c} = -1$, respectively.

We are interested in classifying equivalent projective representations, see [Definition 34](#). To do so, we will use the second cohomology group

$$H^2[G, (\mathbb{R}/2\pi\mathbb{Z}, +)_{\mathbf{c}}] \cong H^2[G, U(1)_{\mathbf{c}}] \tag{D.14}$$

and [Theorem 5](#).

As noted above, quantum states are equivalent up to multiplication by a phase. Hence, we replace the multiplicative factor seen in [Eq. \(D.9\)](#) by $e^{i\alpha(g)}$ for α a 1-cochain. We recall that

$$(\delta_{\mathbf{c}}^1 \alpha)(g, h) = \mathbf{c}(g) \alpha(h) - \alpha(g \cdot h) + \alpha(g). \quad g, h \in G. \tag{D.15}$$

Since 2-cocycles modulo a coboundary are again 2-cocycles, the normalization of the projective representations requires

$$\begin{aligned} 0 &\stackrel{!}{=} (\phi + \delta_{\mathbf{c}}^1 \alpha)(e, e) \\ &= \underbrace{\phi(e, e)}_{=0} + \alpha(e) + \underbrace{\mathbf{c}(e)}_{=1} \alpha(e) - \alpha(ee) \\ &= \alpha(e), \end{aligned} \tag{D.16}$$

where we used that $\mathbf{c}(g)$ is a group homomorphism.

Remark 45. In this context, projective equivalence is nothing else but gauge equivalence in physics.

Remark 46. In light of *Remark 44*, we may identify anti-unitary representations with non-central group extensions.

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